Calculations:

The cantilever system can be modeled as a beam of midline length L, width W(x), and thickness T(x). It is assumed to have uniform density p and Young's Modulus E. The function y(x,t) represents the postition of the cantilever midline along the y-axis, and the cantilever is fixed so that y(0,t)=0 and y'(0,t)=0. The system can be analyzed by considering the forces on any arbitrarily small widthwise section of length dx, and thickness du. The forces in question are tension and shear, F(x) and F(y) respectively. Fx arises from the bending of the material and its bulk elasticity. Fy arises from the fact that the plank must remain a single, continuous piece. Over the bent section indicated by dx,

$$\Delta x = \frac{R+u}{R}dx - dx = \frac{U}{R}dy$$

Where R is given by the second derivative for y(x).

$$y - y_0 = \frac{1}{2}y''dx^2$$

$$R^2 = dx^2 + (y_0 + \frac{1}{2}y''dx^2)^2$$

Because $R = y_0$,

$$0 = dx^2 + Ry''dx^2 + \theta(dx^2)^2$$

$$y'' = -\frac{1}{R}$$

$$R = -\frac{1}{y''}$$

$$\Delta x = -uy''dx$$

For the plank to remain continuous, there must be present both shear (S(x)) and torque. These can be represented as the derivatives in respect to time of each other and y(x).

$$S(x-dx) = S(x) + \frac{d^2y}{dx^2}dm$$

$$\Gamma(x-dx) = \Gamma(x) + S(x)dx$$

This indicates that $S' = -pWT\frac{d^2y}{dt^2}$, $\Gamma' = -S$ and $\Gamma'' = pWT\frac{d^2y}{dt^2}$. By balancing these equations for force we can define an equation from which we can derive a solution.

$$E\frac{d^2}{dx^2}(WT^3\frac{d^2y}{dx^2}) = -pWT\frac{d^2y}{dt^2}$$

This fourth-order differential equation shows similarities to general wave equations, and because of this can support wave-like solutions. A system of conditions is required to solve for a particular solution in all cases; These can be found from the system boundary conditions.

The first scenario assumes the simplest possibility: that the plank width W and thickness T are constant. Simplifying the general equation under these conditions gives

$$\frac{d^4y}{dx^4} = -(\frac{p}{ET^2})\frac{d^2y}{dt^2}$$

Because of the oscillatory properties of the plank's vibration, it is given that

$$y = y_0e^{i(kx-\theta t)}$$

As part of the definition of the eigenmode, y(x,t) can be further partitioned into the product of two seperate one-variable equations.

$$y(x,t) = y_a(x)e^{i\theta t}$$

This allows y to be defined as an eigenfunction in the following equation

$$\frac{d^4y_a}{dx^4} = (\frac{p}{ET^2})\theta^2 y_a$$

Where the fourth derivative is a linear Hermitian operator which returns the function multiplied by a scalar, equivalent to the the constant term. Because it is Hermitian, it is possible to build a general solution for this equation using a linear combination of terms which satisfy certain conditions.

$$y_a = a_1e^{ikx} + a_2e^{-ikx} + a_3e^{kx} + a_4e^{-kx}$$

$$k = (\frac{p\theta^2}{ET^2})^{\frac{1}{4}}$$

Using the four known boundary conditions, a system of equations can be developed to solve for the constant coefficents. The boundary conditions are:

$$y_a(0) = y_a'(0) = 0$$

$$y_a''(L) = y - a'''(L) = 0$$

These make the following system of equations.

$$a_1 + a_2 + a_3 + a_4 = 0$$

$$i(a_1 - a_2) + (a_3 - a_4) = 0$$

$$-a_1e^{ikL} - a_2e^{-ikL} + a_3e^{kL} + a_4e^{-kL} = 0$$

$$-ia_1e^{ikL} + ia_2e^{-ikL} + a_3e^{kL} - a_4e^{-kL} = 0$$

Because this is a homogenous system of equations, there exist solution sets only for specific values of k. Each set represents a different eigenmode in the plank's vibration. It is possible to solve for k by considering the matrix system created by these four equations.

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ ia_1 & -ia_2 & a_3 & -a_4 \\ -e^{ikL} & -e^{-ikL} & e^{kL} & e^{-kL} \\ -ie^{ikL} & ie^{-ikL} & e^{kL} & -e^{-kL} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The determinant of the augmented matrix must be zero, because it is singular. If we let $a = e^i kL$ and $b = e^k L$, then the determinant is:

$$8i + \frac{2i}{ab} + \frac{2ib}{a} + \frac{21a}{b} + 2iab = 0$$

$$4ab + 1 + b^2 + a^2 + a^2b^2 = 0$$

$$(a+b)^2 + (ab+1)^2 = 0$$

There are no possible real solutions for (a, b); a must be complex. If $a = e^{i\omega}$, then:

$$a_r + b = \pm a_i b$$

$$a_r b + 1 = \mp a_i$$

$$a_r = \frac{-2b}{b^2+1}$$

$$a_i = \pm\frac{1-b^2}{1+b^2}$$

Leading to the following identity, which allows a series of k values to be solved for, each of which corresponds to a different eigenmode.

$$\omega = \arctan\frac{b^2-1}{2b} = \pm \ln b$$

$$k = 1.8751, 4.6941, 7.8532, 10.9955, 14.1372, 17.2788, 20.4204, 23.5619, ...$$

Note: Omega here does not mean frequency, as it does elsewhere.

It is now possible to solve for the constants in the solutions to the differential equation by substituting each k value into the homogenous matrix of boundary conditions above and determining the values of a_1, a_2, a_3 and a_4.

For the case of non-uniform width, the values of the constants in the solutions can be found from the original equation. For example, for a diamond shaped cantilever with length equal to its width:

$$\frac{dW}{dx} = \begin{cases} 2 & 0 \le x \le \frac{L}{2} \\ -2 & \frac{L}{2} \le x \le L \end{cases}$$

$$(2x)\frac{d^4y}{dx^4} + (4)\frac{d^3y}{dx^3} - \Omega(2x)y = 0$$

$$\frac{d^4y}{dx^4} + \frac{2}{x}\frac{d^3y}{dx^3} - \Omega y = 0$$

This equation can by symbolized using the following operators:

$$H = H_0 + H'$$

And can be solved for the necessary constants using the following steps.

$$y_i^{(4)} = \Omega\omega^2 y_i$$

$$H_0 y_i = e_i y_i$$

$$H = H_0 + H'$$

$$Hy(x) = H\sum_i c_i y_i(x) = \epsilon y(x)$$

$$\sum_i c_i (H y_i)$$

$$\sum_i c_i (H_0 y_i + H' y_i)$$

$$\sum_i c_i \epsilon y_i = \sum_i c_i e_i y_i + \sum_i c_i H' y_i$$

$$(y_j, y_i) = \delta_{ij}$$

$$(y_j, H' y_i) = H'_{ij}$$

$$c_j \epsilon = c_j e_j + \sum_i c_i H'_{ij}$$

$$H'_{ij} c_i = c_j(\epsilon - e_j)$$

$$(H'_{ij} + e_j \delta_{ij}) c_i = \epsilon c_j$$

$$(H'_{ji} + e_i \delta_{ji}) c_j = \epsilon c_j$$