**Preliminary Exam: Statistical Mechanics, Tuesday August 20, 2019. 9:00-12:00**

Answer a total of any **THREE** out of the four questions. Put the solution to each problem in a **SEPARATE** blue book and put the number of the problem and your name on the front of each book. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded. **Some possibly useful information**:

$$\text{Sterling's asymptotic series}: \quad \ln N! \approx N \ln N - N + \frac{1}{2}\ln[2\pi N] \quad \text{as} \quad N \to \infty,$$

$$\int_0^\infty dx\ x\ \exp(-\alpha x^2) = \frac{1}{2\alpha}, \qquad \int_{-\infty}^{+\infty} dx\ \exp(-\alpha x^2\ +\ \beta x) = \sqrt{\frac{\pi}{\alpha}}\exp(\frac{\beta^2}{4\alpha})\ \text{with}\ \mathrm{Re}(\alpha) > 0.$$

1. (**a**) In quantum statistical mechanics the thermal density operator of the canonical ensemble may be derived by finding the density operator $\rho$ that minimizes the Boltzmann entropy $S = -k_B \mathrm{Tr}(\rho \ln \rho)$ under the constraint that the expectation value of the energy $\mathrm{Tr}(\rho H)$ has a fixed value. ($k_B$ is Boltzmann's constant.) If the system also conserves total angular momentum, one imposes the additional constraint that $\mathrm{Tr}(\rho \mathbf{L})$ is fixed. Show that the thermal density operator in this case is

$$\rho = \frac{1}{Z} e^{-\beta(H - \boldsymbol{\omega}\cdot\mathbf{L})}, \quad Z = \mathrm{Tr}\, e^{-\beta(H-\boldsymbol{\omega}\cdot\mathbf{L})},$$

where $\beta$ and $\boldsymbol{\omega}$ are so far undetermined Lagrange multipliers. By comparing with thermodynamics, one finds the usual identification $\beta = 1/k_B T$.

(**b**) Suppose we are dealing with the classical limit of statistical mechanics, for an ideal gas of particles each with mass $m$ in an isotropic harmonic-oscillator potential $V(\mathbf{r}) = \frac{1}{2} m\omega_0^2 \mathbf{r}^2$. In thermal equilibrium, the prescription of part **(a)** leads to the joint probability density function (PDF) for position $\mathbf{r}$ and velocity $\mathbf{v}$ ($= \mathbf{p}/m$) of each particle

$$f(\mathbf{r}, \mathbf{v}) = K \exp\left[-\frac{m}{k_B T}\left(\frac{1}{2}\mathbf{v}^2 + \frac{1}{2}\omega_0^2 \mathbf{r}^2 - \boldsymbol{\omega}\cdot\mathbf{r}\times\mathbf{v}\right)\right],$$

where $K$ is a normalization constant. Show that at a given position $\mathbf{r}$ the flow velocity (mean or average velocity) of the gas is

$$\langle \mathbf{v}(\mathbf{r})\rangle = \boldsymbol{\omega}\times\mathbf{r}\,.$$

(**c**) What is the physical meaning of the vector $\boldsymbol{\omega}$ in part **(b)**?

(**d**) BONUS. We have to assume that $|\boldsymbol{\omega}| < \omega_0$, otherwise the presumed PDF is not normalizable and the argument fails. What would be the *physical* reason for the failure?

2. The volume of a sphere of radius $R$ in $D > 0$ dimensions is

$$V_D(R) = \frac{\pi^{D/2}}{\Gamma(D/2+1)}\, R^D,$$

where $\Gamma$ is the gamma function. Consider an ideal Fermi gas with degeneracy $g$ and dispersion relation of the particles $\epsilon(k) = Ck^\alpha$, where $k$ is the wave number, and $\alpha > 0$ and $C > 0$ are constants.

(**a**) Consider a two-dimensional system. Assume that the motion of a free particle is quantized in a square box of area $A$ with periodic boundary conditions. Write down the volume in the $k$ space associated with each eigenstate. What is the corresponding density of states in $k$ space?

(**b**) Now consider an arbitrary number of dimensions $D$. Show that the Fermi energy (chemical potential at $T = 0$) as a function of the $D$-dimensional particle density $n$ (number of particles, regardless of spin, per unit $D$-dimensional volume) is

$$\epsilon_F = C\left[2\sqrt{\pi}\left(\frac{n\,\Gamma(1+D/2)}{g}\right)^{1/D}\right]^\alpha$$

(**c**) What would be the values of the constants $C$, $\alpha$ and $g$ for massive spin-$\frac{1}{2}$ particles? Given that $\Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}$, verify that in 3D the result of part **(b)** agrees with the standard Fermi energy expression

$$\epsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}.$$

3. A two-dimensional (2D) ideal gas of spinless Bose particles occupies a large area $A$. The dispersion law $\epsilon(\mathbf{p}) = \alpha p$ describes the relation between the particle energy $\epsilon$ and momentum $\mathbf{p}$, where $\alpha$ is a positive constant. The number of particles $N_g(\mu, T, A)$ in the gas is a function of the chemical potential $\mu$, the temperature $T$, and the area $A$.

   (**a**) Calculate the statistical sum $Z$ and free energy $F = -k_B T\, lnZ$ for an arbitrary gas temperature $T$, if the chemical potential is fixed at zero value: $\mu = 0$.

   (**b**) Determine the total energy $E$, the entropy $S$, and the total number of particles $N_g(\mu = 0, T, A)$ in this Bose gas.

   (**c**) What role does the particular value of the particle spin $s = 0, 1, 2...$ play in determining your answers to parts **(a)** and (**b**)? Justify your answer.

   Hint: Integrals required for solution of this problem

$$\int_0^\infty \frac{x^{k-1}}{\exp(x) - 1} dx = \Gamma(k)\zeta(k), \qquad \Gamma(k) = \int_0^\infty e^{-t} t^{k-1} dt, \quad \zeta(k) = \sum_{n=1}^{n=\infty} n^{-k}$$

   can be expressed in terms of the gamma function $\Gamma(k)$ and Riemann $\zeta(k)$ functions for all $k > 1$.

4. A classical gas of atoms each of mass $m$ is contained in a cylindrical atomic trap with length $L$ and radius $R$ where $L \gg R$. The thermal equilibrium of the trapped gas is permanently supported at constant temperature $T$ .

   (**a**) Calculate the time dependent flux $\Phi(t)$ of atoms escaping from the trap, if the number of atoms $N(t)$ inside the trap is known at t=0: $N(t = 0) = N_0$. The side surface of the cylinder is impenetrable and atoms can escape only through the two ends of the cylinder. The probability of reflection from the base surface is $p$.

   (**b**) Determine the value of $\Phi(t)$ and $N(t)$, if the reflection probability depends on the z-component of the atomic velocity $v_z$: $p(v_z) = \exp(-\alpha v_z^2)$, where $z$ is the axis of symmetry of the cylinder and $\alpha$ is a positive constant. Calculate the average energy $< \epsilon >$ of atomic particles in the escape flux.

   (**c**) How would the results for $\Phi(t)$, $N(t)$, and $< \epsilon >$ change if the ends of the cylinder were impenetrable and the side surface was assigned to be penetrable with the same reflection probability $p(v_z)$ defined in part **(b)**?