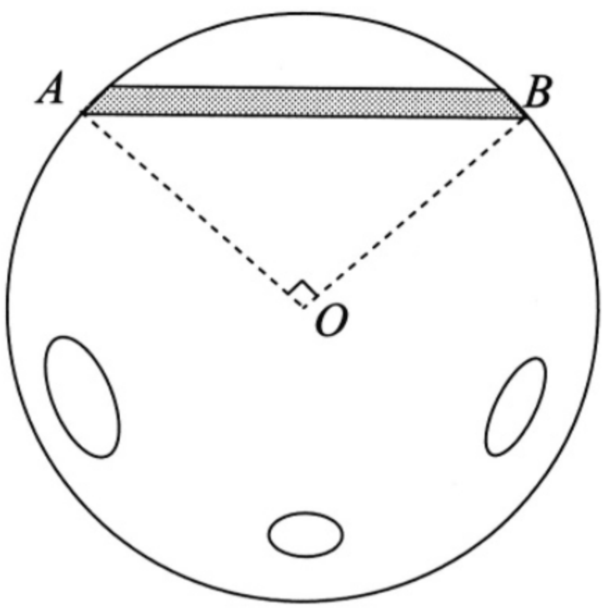
Preliminary Exam: Statistical Mechanics, Tuesday August 22, 2016. 9:00-12:00

Answer a total of any **THREE** out of the four questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded. **Some possibly useful information**:

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty, \qquad \int_0^\infty dx\; x\; \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

$$\int_{-\infty}^{+\infty} dx\; \exp(-\alpha x^2 \;+\; \beta x) = \sqrt{\frac{\pi}{\alpha}}\exp(\frac{\beta^2}{4\alpha}) \text{ with } \mathrm{Re}(\alpha) > 0$$

1. A well-known science fiction novel describes an encounter of amateur earthling astronauts with an expatriate human civilization living in very deep caverns beneath the surface of a spherical planet. The caverns are connected to the surface by long channels filled with air. A channel is dug between points A and B on the surface of the planet so that the angle AOB is 90^o where O is the center of the planet (see figure). The acceleration due to gravity on the surface of the planet is a fifth of that at the surface of the Earth.



(a) Take the pressure at the surface of the planet to be P_1. Derive an analytic expression that gives the air pressure P_0 at the midpoint of the channel.

Hint: Consider the atmosphere to be isothermal inside the channel. Assume that the planet is made from a solid with constant density, that the atmosphere, as much as it exists on the planet, is pure oxygen. Assume that the tunnels are so small that they have no effect on the gravitational field of the planet.

(b) Give a numerical order of magnitude estimate of the pressure at the midpoint of the AB channel if the surface air pressure of the planet is 10^{-4} atmospheres, and compare it to the one atmosphere pressure at the surface of the Earth. The mass of an oxygen molecule divided by the Boltzmann constant is given by $m(O_2)/k_B = 1.92 \cdot 10^{-3}$ m^{-2} s^2 K. Assume a constant temperature of $300^o K$ on the planet surface and inside the tunnels. The radius of the planet is $r = 2500$ km.

2. Calculate the second virial coefficient for the two cases:

   (i) the classical hard sphere gas.

   (ii) non-interacting fermions.

   Hint: The virial coefficients are defined via

$$\frac{P}{k_B T} = \sum_{N=1}^{\infty} B_N(T) n^N,$$

   with particle number N, pressure P, temperature T, relative particle number $n = N/V$. The first two coefficients are of the form

$$B_1(T) = 1, \qquad B_2(T) = \frac{\tilde{Q}_2(T)}{2V},$$

   where

$$\tilde{Q}_2 = Q_2 - Q_1^2, \qquad Q_1 = \int\limits_V d^3x = V, \qquad Q_2 = \iint\limits_V d^3x_1 d^3x_2\,(1 + f_{12}).$$

   The interaction term f_{12} is defined via $f_{12} = e^{-\beta U_{12}} - 1$, where U_{12} is the interaction energy between particle 1 and particle 2, and $\beta = 1/k_B T$. In a classical hard sphere gas the gas molecules consist of hard spheres each of radius R, i.e. the interaction goes to infinity whenever the spheres overlap, i.e. when the centers of the spheres are closer than $2R$. For the Fermi gas while full credit requires determining $B_2(T)$, you can get partial credit for pinpointing what the second virial coefficient depends on.

3. An ideal gas of N fermions each with the spin $s = 1/2$ occupies a 3-dimensional volume V. Particles in the gas are enclosed by impenetrable walls, and E_F is the Fermi energy of the gas. Calculate the average rate (frequency) f of fermion collisions with the walls as normalized per unit wall area for the following three cases:

   (a) the high temperature classical gas for which $k_B T \gg E_F$, with the particle energy and momentum obeying $E_p = p^2/2m$

   (b) the degenerate Fermi gas for which $k_B T \ll E_F$, with the particle energy and momentum obeying $E_p = p^2/2m$.

   (c) the ultra-relativistic gas for which $k_B T \gg E_F$, and $E_p = cp$.

4. The vibrational spectrum of a diatomic molecule consisting of unlike atoms is described by a harmonic oscillator with the energy levels given by the usual $E_n = (n+1/2)\hbar\omega$, where $n = 0, 1, 2, 3, \ldots$. Consider an ensemble of N noninteracting diatomic molecules at temperature T.

   (a) Calculate the partition function Z_{vib}, the entropy S_{vib}, and free energy F_{vib} corresponding to vibrational motion.

   (b) Determine an asymptotic expression for the free energy F_{vib} at high temperature $k_B T \gg \hbar\omega$ and at low temperature $k_B T \ll \hbar\omega$.

   (c) Calculate the rotational partition function Z_{rot}, the rotational entropy S_{rot}, and the free energy F_{rot} if the quasi-classical angular momentum $\mathbf{J}$ has only two projections J_x and J_y, both of which are perpendicular to the molecular axis. Take the moment of inertia for the x and y axes to be the same: $I_x = I_y = I$. The rotational energy E_{rot} is given by:

$$E_{\text{rot}} = \frac{\hbar^2}{2I}\left(J_x^2 + J_y^2\right).$$

   Hint: In this calculation $k_B T \gg \hbar^2/2I$.