Preliminary Exam: Quantum Mechanics, Friday August 24, 2018. 9:00am-1:00pm

Answer a total of any **FOUR** out of the five questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than four problems, only the first four problems as listed on the exam will be graded.

Some possibly useful information:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2},\ H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2-2.$$

$$\text{Laguerre} = L_n(r) = e^r\frac{d^n}{dr^n}\left(r^n e^{-r}\right),\ \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q\frac{d^q}{dr^q}L_{n+q}(r).$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l,\ P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2-1),$$

$$\int_{-1}^{+1} dw P_\ell(w)P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}.$$

$$\text{associated Legendre polynomial} = P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} = Y_l^m(\theta,\ \phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}\ ,$$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}\ ,\ Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta\ ,\ Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta-1)\ ,\ Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi}\ ,\ Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels}:\quad j_l(r) = (-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right)\ ,\ n_l(r) = (-1)^{(\ell+1)}r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right)\ ,$$

$$\text{with asymptotic behavior}\quad j_\ell(r)\to\frac{\cos(r-\ell\pi/2-\pi/2)}{r}\ ,\ n_\ell(r)\to\frac{\sin(r-\ell\pi/2-\pi/2)}{r}\ .$$

$$j_0(r) = \frac{\sin r}{r}\ ,\ n_0(r) = -\frac{\cos r}{r}\ ,\ j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r}\ ,\ n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r}\ ,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2}\ ,\ n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2}\ .$$

1. Consider a non-relativistic spin-zero particle of mass m in the potential $V(\vec{r}) = \frac{c}{r^2} + \frac{1}{2}m\omega^2 r^2$, $c > 0$.

   a) Without detailed calculations determine the general properties of the energy spectrum you expect for the Hamilton operator of this system: discrete or continuous, non-degenerate or degenerate, and if degenerate, why and what is the degree of degeneracy?

   b) Let $\psi(\vec{r})$ denote the wave-function of the particle, and let $u(r)$ be the radial function defined by the Ansatz $\psi(\vec{r}) = \frac{u(r)}{r} \times$ (angular parts) where $r = |\vec{r}|$. Derive the radial equation for $u(r)$. Determine the small-r behavior and the large-r behavior of $u(r)$.

   c) Starting with the Ansatz $u(r) = r^A \exp(-Br^2)\, g(r)$ with A and B known from part (b), take the function $g(r) = d_0 =$ constant for the ground state. From this information determine the ground state energy E_0. Show that you recover a well-known result if you take the limit $c \to 0$ in the potential $V(\vec{r})$.

2. Consider a particle of mass m in one dimension subject to the double delta function potential,

$$V(x) = -g\delta(x+a) - g\delta(x-a)$$

   where g is a positive constant.

   a) Write down the general form of the solution to the Schrödinger equation for this potential in the three separate regions, (i) $x < -a$, (ii) $-a \leq x < a$, and (iii) $a \leq x$.

   b) Find an equation which determines the energy of the lowest bound state. Sketch a graph that shows how many solutions exist to this equation.

   c) Using this equation, find approximate expressions for the ground state energy in the limits where $a \to 0$ and $a \to \infty$.

   d) How many bound states are there for this potential? If your answer depends on the value of $g > 0$, what are the critical values of g where the answer changes?

3. Consider the two potential energy functions,

$$\begin{aligned} V_1(x) &= \begin{cases} \frac{1}{2}m\omega^2 x^2 & : \ x > 0, \\ \infty & : \ x \leq 0, \end{cases} \\ V_2(x) &= \quad \frac{1}{2}m\omega^2 x^2 \ : \ \text{for all } x. \end{aligned}$$

A particle of mass m is initially subject to potential $V_1(x)$, and is in its ground state.

a) What is the ground state energy E_1 of potential $V_1(x)$? Answer the same question for the ground state energy E_0 of $V_2(x)$. You should be able to do this immediately by inspection, without doing any actual work.

b) Write down the normalized wavefunction for the initial state, which is the ground state of V_1.

c) At $t = 0$ the impenetrable potential barrier in V_1 at $x = 0$ is suddenly removed, so that for $t > 0$ the system is subject to potential V_2 instead of V_1. If the energy is measured some time later, what is the probability that the measurement will yield E_0?

4. Consider a Hermitian Hamiltonian $H(\lambda)$ with the energies $E_n(\lambda)$ and the corresponding normalized eigenstates $|n(\lambda)\rangle$, which all depend smoothly on the parameter λ.

a) Show that the normalization condition for the states implies

$$\left(\frac{d}{d\lambda}\langle n|\right)|n\rangle + \langle n|\left(\frac{d}{d\lambda}|n\rangle\right) = 0\,.$$

b) Prove the Feynman-Hellman theorem

$$\frac{dE_n}{d\lambda} = \left\langle n \left| \frac{dH}{d\lambda} \right| n \right\rangle.$$

Now a little application of the Feynman-Hellman theorem:

(c) Prove the equipartition theorem for the harmonic oscillator potential, which says that expectation values of kinetic and potential energies are equal in the energy eigenstates of the harmonic oscillator Hamiltonian.
HINT: Regard the mass as a variable parameter.

(d) Use the same technique to derive a relation between $\langle T\rangle$ and $\langle V\rangle$ for the hydrogen atom in the energy eigenstate $|nlm\rangle$.

5. Consider a two-level atom with the state space spanned by the two orthonormal states $|1\rangle$ and $|2\rangle$, dipole-coupled to an external time dependent driving field $E(t)$. After a unitary transformation to the "rotating frame", the Hamiltonian reads

$$\frac{H}{\hbar} = \Delta\,|2\rangle\langle 2| - f(t)\Big(\,|2\rangle\langle 1| + |1\rangle\langle 2|\,\Big)$$

where Δ is the difference between the atomic transition frequency ω_0 and the frequency of the nearly monochromatic driving field ω, and $f(t)$ is proportional to the temporal envelope of the driving field: $E(t) \propto f(t)\cos(\omega t)$.

Suppose that

$$f(t) = \frac{\lambda}{2\sqrt{\pi}\,\tau}\left\{\exp\left[-\left(\frac{t+T/2}{\tau}\right)^2\right] + \exp\left[-\left(\frac{t-T/2}{\tau}\right)^2\right]\right\}.$$

This represents two pulses of light of length τ hitting the system at times $\mp T/2$. Let us assume that the amplitude of the driving field $\propto \lambda$ is "very small", and that the system starts out in the state $|1\rangle$.

a) Expand a general wavefunction of the system in the basis of stationary states $|1\rangle$ and $|2\rangle$ of the unperturbed atomic Hamiltonian, and write the solutions for the two expansion coefficients $c_1(t)$ and $c_2(t)$ to leading order in λ.

b) Find the probability that the system is in the state $|2\rangle$ after the pulses are gone. NOTE: You are describing what is called *Ramsey fringes*, the foundation of modern ultrahigh-precision spectroscopy.