**QUANTUM MECHANICS**

# Preliminary Examination

Friday 08/22/2014

09:00–13:00 in P-121

Answer a total of **FOUR** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on sheets of paper stapled together. Make sure you clearly indicate who you are and what is the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

You are allowed to use a result stated in one part of a problem in the subsequent parts even if you cannot derive it. On the last page you will find some potentially useful formulas.

**Problem 1.** Let us define $D = \frac{1}{2}(xp + px)$, where $x$ is the position operator and $p$ the momentum operator in one dimension.

(a) Calculate $[D, x^m]$ and $[D, p^n]$ where $m$ and $n$ are integers.

(b) Consider the Hamilton operator $H = \dfrac{p^2}{2m} + V(x)$ with the potential $V(x) = \alpha\, x^\beta$ where $\alpha$ and $\beta$ are real non-zero constants. Calculate $U(\lambda) H U^\dagger(\lambda)$ with $U(\lambda) = \exp(i\lambda D/\hbar)$.

(c) There is a value for $\beta$ in the potential $V(x) = \alpha\, x^\beta$ for which the Hamiltonian in part (b) transforms as $U(\lambda) H U^\dagger(\lambda) = f(\lambda)\, H$. What is the function $f(\lambda)$?

Hints: Recall the identity for two non-commuting linear operators $A$ and $B$

$$\exp(\lambda A)\, B \exp(-\lambda A) = B + \frac{\lambda^1}{1!}[A, B] + \frac{\lambda^2}{2!}[A, [A, B]] + \frac{\lambda^3}{3!}[A, [A, [A, B]]] + \ldots .$$

You may do the mathematics formally, ignoring issues such as the precise definitions and domains of various operators.

**Problem 2.** (a) Given the usual eigenstates $|j, m\rangle$ of the angular momentum operators $\mathbf{J}^2$ and $J_z$, determine the expectation values $\langle j, m|J_x|j, m\rangle$ and $\langle j, m|J_y|j, m\rangle$.

(b) Find the standard deviation $\Delta J_x = \sqrt{\langle j, m|J_x^2|j, m\rangle - \langle j, m|J_x|j, m\rangle^2}$, and $\Delta J_y$ defined analogously.

(c) Determine the eigenvalues and construct the *real* eigenfunctions of the Hamiltonian involving orbital angular momentum of a single particle, $H = a(L_x^2 + L_y^2) + bL_z^2$, where $a \neq b$ are real constants.
Possibly helpful identity: $Y^*_{l,m}(\theta, \phi) = (-1)^m\, Y_{l,-m}(\theta, \phi)$.

**Problem 3.** A particle of mass $m$ and electric charge $q$ is constrained to move in a tightly confining ring of radius $R$; call the remaining coordinate along the ring $x$. The motion along $x$ is free, i.e., there are no forces acting on the particle in the direction $x$. Determine:

(a) Eigenvalues and eigenfunctions of energy.

(b) The maximum value of the electric current $I$ in the first excited state.

Hint: The current density of a quantum particle is $\mathbf{j} = \dfrac{i\hbar q}{2m}(\psi \nabla \psi^* - \psi^* \nabla \psi)$.

**Problem 4.** A particle with the energy $E$ and mass $m$ is scattered by the potential field $U(r) = U_0(R/r)\exp(-r/R)$, where $U_0$ and $R$ are positive constants. Calculate the scattering amplitude $f(E, \theta)$ ($\theta$ is the scattering angle) and the total scattering cross section $\sigma$ using the first Born approximation.

**Problem 5.** Consider the Hamiltonian

$$H = E_1\,|1\rangle\langle 1| + E_2\,|2\rangle\langle 2| + V\,|2\rangle\langle 1| + V^*|1\rangle\langle 2|\,,$$

with $|V| \ll |E_2 - E_1|$.

(a) Find the eigenvalues of energy and the corresponding normalized eigenstates up to the lowest nontrivial order in the strength of the perturbation $V$. Denote these by $E_1'$, $|1'\rangle$ and $E_2'$, $|2'\rangle$, with $E_1' \to E_1$ as $V \to 0$ and so on.

(b) Suppose we are studying transitions from yet another state $|g\rangle$ to the states $|1\rangle$ and $|2\rangle$ governed by the operator $D$, and have the known transition matrix elements $\langle 1|D|g\rangle = d$, $\langle 2|D|g\rangle = 0$. At this level the transition $g \to 2$ is evidently forbidden. However, the perturbation $V$ leads to a small admixture of the original state $|1\rangle$ in the state $|2'\rangle$. Thus, a transition that to an observer unaware of the existence of perturbation $V$ might seem to be $g \to 2$ is possible after all. Find the corresponding matrix element $\langle 2'|D|g\rangle$.

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty$$

$$\int_{-\infty}^{+\infty} dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \quad \text{with} \quad \mathrm{Re}(\alpha) > 0$$

$$\int_0^{\infty} dx\; x \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i \frac{p}{m\omega}\right)$$