Preliminary Examination: Quantum Mechanics, 08/24/2012

Answer a total of **FOUR** questions out of **FIVE**. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book, or on consecutively numbered sheets of paper stapled together. Make sure you clearly indicate who you are, and the problem you are solving. Double-check that you include everything you want graded, and nothing else.

Possibly Useful Information

$$\int_{-\infty}^{+\infty} dx\ \exp(-\alpha x^2\ +\ \beta x) = \sqrt{\frac{\pi}{\alpha}}\exp(\frac{\beta^2}{4\alpha}) \text{ with } \mathrm{Re}(\alpha) > 0, \qquad \int_0^{\infty} dx\ x\ \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

First few spherical harmonics

$$Y_{00} = \left(\frac{1}{4\pi}\right)^{1/2}, \quad Y_{10} = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta, \quad Y_{1\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\ \exp(\pm i\phi)$$

$$Y_{20} = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1), \quad Y_{2\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta\ \exp(\pm i\phi),$$

$$Y_{2\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta\ \exp(\pm 2i\phi)$$

Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1)$$

1. Dirac notation is cumbersome in problems involving complex and hermitian conjugates, so let us presently denote the inner product of two states ψ and ϕ by (ψ, ϕ).

   a) Show that an arbitrary linear operator Q may always be written in the form $Q = A + iB$, where A and B are both hermitian operators.

   b) Suppose that $(\psi, Q\psi) = 0$ for an arbitrary state ψ. Then the same also holds true for the state $\Psi = \psi + \lambda\phi$, no matter what the states ψ and ϕ and the scalar λ might be. By first picking $\lambda = 1$ and then $\lambda = i$, show that $(\psi, Q\phi)$=0 for all states ψ and ϕ.

   c) Show that if $(\psi, Q\psi) = 0$ for an arbitrary state ψ, then Q is in fact the zero operator that maps all states to the zero vector.

   d) Take it as given that the expectation value of a hermitian operator A in any state ψ is real. Based on the stated results of parts (a) and (c), show that the converse also holds true, that an operator Q whose expectation value is real in all states ψ must be hermitian.

   Hint: You may freely use a result stated in one part of the problem in the other parts, even if you cannot derive it.

2. Consider the relativistic expression for kinetic energy

$$E(p) = \sqrt{p^2c^2 + (mc^2)^2} \simeq mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}$$

   If a particle like this is bound to a harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$ the oscillator is no longer exactly harmonic, and the transition frequency between two adjacent states $\Omega_n = (E_{n+1} - E_n)/\hbar$ will depend on the index n. The effect is small for non-relativistic oscillators, but with the fantastic precision of spectroscopic techniques it is occasionally detectable even when the oscillator is near its ground state. Find Ω_n in the limit of weak nonlinearity.

   Hint: The form of the number operator for the simple harmonic oscillator is $N = a^\dagger a$, where

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right)$$

   To reduce the tedium, you may want to use units in which $m = \omega = \hbar = 1$

3. Consider a one-electron "atom" with the unperturbed Hamiltonian H_0

$$H_0 = \frac{1}{2m}p^2 + V(r)$$

This electron with charge q is also subjected to external electromagnetic radiation. In the electric dipole approximation we ignore both the variation of the electric field across the atom and the magnetic field, so that the vector and scalar potentials may be chosen to be of the form $\vec{A}(\vec{r},t) = \vec{A}(t)$, $\phi = 0$. Now, with certain canonical transformations, the usual minimum-coupling Hamiltonian gives two different forms for the atom-field interaction

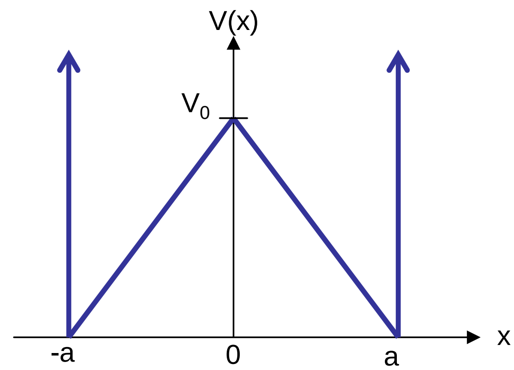
$$H'_{d\cdot E} = -q\vec{r}\cdot\vec{E}(t)$$
$$H'_{p\cdot A} = -\frac{q}{m}\vec{p}\cdot\vec{A}(t)$$

a) By studying the commutator $[x, H_0]$ show that the matrix elements of position and momentum between the eigenstates $|n\rangle$ of H_0 satisfy $\langle\ell|\,\vec{p}\,|n\rangle = im\omega_{\ell n}\,\langle\ell|\,\vec{r}\,|n\rangle$, where $\omega_{\ell n} = (E_\ell - E_n)/\hbar$ is the frequency difference between the states $|\ell\rangle$ and $|n\rangle$.

b) Both $H'_{d\cdot E}$ and $H'_{p\cdot A}$ are used in the analysis of light-matter interactions. This gives rise to the alarming possibility that the calculation of the same quantity could give two different results depending on which form of the interaction is used. However, this does not happen frequently. Suppose the atom is driven by electromagnetic fields such that $\vec{A}(t)$ tends to zero smoothly with $t \to \pm\infty$. Consider transitions from some initial ($t = -\infty$) state of the atom $|n\rangle$ to the other states using first-order time dependent perturbation theory. Show that the transition probabilities after the fields have turned back to zero ($t = \infty$), are the same for the two choices of the interaction Hamiltonian.

4. A particle of mass m obeying the 1D Schrodinger equation moves in the potential $V(x)$ depicted in the figure at the right.

$$V(x) = \begin{cases} V_0 \left(\frac{a-|x|}{a}\right) & : \quad |x| < a \\ \infty & : \quad |x| \geq a \end{cases}$$



a) In the limit $V_0 \to 0$ with fixed a, what are the energy eigenvalues and eigenfunctions of the system?

b) In the limit $a \to 0$ with fixed V_0, what are the energy eigenvalues and eigenfunctions of the system? Express the answers in terms of a small but non-zero value for a.

c) What are the energy eigenvalues and eigenfunctions of the system for the general case of arbitrary V_0 and a?

You should label the zeros of $Ai(z)$ as $\alpha_1, \alpha_2, \ldots$ and of its first derivative as $\alpha'_1, \alpha'_2, \ldots$ in order of increasing magnitude. Likewise the zeros of $Bi(z)$ should be written as $\beta_1, \beta_2, \ldots$ and of its first derivative as $\beta'_1, \beta'_2, \ldots$. You may find the following properties of the Airy functions Ai and Bi to be useful in solving this problem.

$$Ai(z) \to \begin{cases} \frac{1}{\sqrt{\pi}}\, z^{-1/4}\, \cos(\xi - \frac{\pi}{4}) & : \quad z \ll 0 \\ \frac{1}{2\sqrt{\pi}}\, z^{-1/4}\, e^{-\xi} & : \quad z \gg 0 \end{cases}$$

$$Bi(z) \to \begin{cases} -\frac{1}{\sqrt{\pi}}\, z^{-1/4}\, \sin(\xi - \frac{\pi}{4}) & : \quad z \ll 0 \\ \frac{1}{\sqrt{\pi}}\, z^{-1/4}\, e^{\xi} & : \quad z \gg 0 \end{cases}$$

$$\xi = \frac{2}{3}|z|^{3/2}, \qquad z = \left[\frac{2mV_0}{\hbar^2 a}\right]^{1/3} (a_0 - |x|)$$

5. A beam of massive spin-1 particles passes through a Stern-Gerlach apparatus and splits into three output beams, each one corresponding to one of the allowed projections of the particle spins onto a direction defined by the magnetic field inside the apparatus. Unless stated otherwise, all directions in this problem are with respect to a fixed laboratory coordinate system.

a) Using the standard raising and lowering operators of angular momentum $J_\pm = J_x \pm i J_y$ such that $J_\pm |jm\rangle = \sqrt{j(j+1) - m(m \pm 1)}\, |jm \pm 1\rangle$, show that the three matrices below form a valid representation of the spin operators for these particles.

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

b) Suppose the incoming beam is initially completely unpolarized, so its density operator in the matrix representation given in part (a) is

$$\rho_0 = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This density operator may be written in terms of the three eigenstates of the spin in the z direction as

$$\rho_0 = \frac{1}{3} \left(|+\rangle \langle +| \; + \; |0\rangle \langle 0| \; + \; |-\rangle \langle -| \right)$$

not only in the laboratory frame, but in any Cartesian coordinate system obtained from the laboratory frame with an arbitrary rotation. Why?

c) What fraction of the particles in the unpolarized beam will pass through a Stern-Gerlach filter set up in such a way that only the particles with the component of the spin equal to 0 in the x direction are selected?