Preliminary Exam: Quantum Mechanics, Friday January 13, 2017. 9:00-1:00

Answer a total of any **FOUR** out of the five questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than four problems, only the first four problems as listed on the exam will be graded.

Some possibly useful information

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$
$$= \frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial z^2}$$
$$\nabla\psi = \mathbf{e}_x\frac{\partial\psi}{\partial x}+\mathbf{e}_y\frac{\partial\psi}{\partial y}+\mathbf{e}_z\frac{\partial\psi}{\partial z} = \mathbf{e}_r\frac{\partial\psi}{\partial r}+\mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta}+\mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} = \mathbf{e}_\rho\frac{\partial\psi}{\partial\rho}+\mathbf{e}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}+\mathbf{e}_z\frac{\partial\psi}{\partial z}.$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}\ ,\ H_0(x)=1\ ,\ H_1(x)=2x\ ,\ H_2(x)=4x^2-2$$

$$\text{Laguerre} = L_n(r) = e^r\frac{d^n}{dr^n}\left(r^n e^{-r}\right)\ ,\ \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q\frac{d^q}{dr^q}L_{n+q}(r)\ .$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l\ ,\ P_0(x)=1\ ,\ P_1(x)=x\ ,\ P_2(x)=\frac{1}{2}(3x^2-1)\ ,$$
$$\int_{-1}^{+1} dw P_\ell(w)P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} = P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} = Y_l^m(\theta,\ \phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2}P_l^m(\cos\theta)e^{im\phi}\ ,$$

$$Y_0^0=\left(\frac{1}{4\pi}\right)^{1/2}\ ,\ Y_1^0=\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta\ ,\ Y_1^{\pm1}=\mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0=\left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta-1)\ ,\ Y_2^{\pm1}=\mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi}\ ,\ Y_2^{\pm2}=\left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels}:\quad j_l(r) = (-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right)\ ,\ n_l(r) = (-1)^{(\ell+1)}r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right)\ ,$$

$$\text{with asymptotic behavior}\quad j_\ell(r)\to\frac{\cos(r-\ell\pi/2-\pi/2)}{r}\ ,\ n_\ell(r)\to\frac{\sin(r-\ell\pi/2-\pi/2)}{r}\ .$$

$$j_0(r)=\frac{\sin r}{r}\ ,\ n_0(r)=-\frac{\cos r}{r}\ ,\ j_1(r)=\frac{\sin r}{r^2}-\frac{\cos r}{r}\ ,\ n_1(r)=-\frac{\cos r}{r^2}-\frac{\sin r}{r}\ ,$$
$$j_2(r)=\frac{3\sin r}{r^3}-\frac{\sin r}{r}-\frac{3\cos r}{r^2}\ ,\ n_2(r)=-\frac{3\cos r}{r^3}+\frac{\cos r}{r}-\frac{3\sin r}{r^2}\ .$$

1. A particle of the mass m is bound by the asymmetric one-dimensional potential $V(x)$:

$$V(x \le 0) = \infty, \qquad V(0 < x \le a) = -U, \quad V(x > a) = 0$$

where U is a positive constant.

This potential can support a weakly bound state with an eigenenergy E, which is much smaller than the depth of the potential: $|E|/U \ll 1$.

(**a**) Derive an exact transcendental equation for the determination of the spectrum of discrete eigenenergies, and show that it can be written in the form:

$$\sin(ka) = \pm \left(\frac{\hbar^2}{2ma^2U} \right)^{1/2} ka, \quad \text{where} \quad k = \left(\frac{2m(U+E)}{\hbar^2} \right)^{1/2}$$

(**b**) Find a threshold value of the potential depth $U = U_C$ for which the first discrete level occurs.

(**c**) Calculate the energy of the first weakly bound state as a function of $\delta U = U - U_C$, if $\delta U/U_C \ll 1$.

Hint: Expand the left and right sides of the equation given in part (**a**) with respect to small parameters $\delta U/U_C \ll 1$ and $|E|/U_C \ll 1$ and keep the leading terms. This then yields a relation between E and δU.

2. The Hamiltonian operator $\hat{H}$ of a 2-dimensional charged particle moving in the (x, y) plane in a uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$ is given by the formula:

$$\hat{H} = \frac{1}{2m} \left(\hat{\mathbf{p}} - e\mathbf{A} \right)^2$$

where $\mathbf{e}_z$ is a unit vector pointing in the z-direction, e and m are the particle's charge and mass, c is speed of light, and $\mathbf{A}$ is the vector potential. The components of the vector potential are given by: $A_x = -By$, $A_y = 0$, $A_z = 0$.

(**a**) Write down the Schrödinger equation for this system.

(**b**) Show that operator $\hat{\mathbf{p}}_x$ commutes with the Hamiltonian, and show that the particle wave function can be expressed as $\Psi(x, y) = \exp(ip_x x/\hbar) Y_{p_x}(y)$ where p_x is an integral of the motion and $Y_{p_x}(y)$ is a function of y.

(**c**) Determine the particle's eigenenergies and wave functions, using the known solutions to the one-dimensional harmonic oscillator.

(**d**) Explain why the particle's energy does not depend on p_x, which is an integral of the motion.

3. The Hamiltonian of a free 2-dimensional harmonic oscillator moving in the (x, y) plane is given as

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2)$$

(**a**) What are the energy eigenfunctions and eigenvalues of this system, and what is their degeneracy?

(**b**) The oscillator is perturbed by an interaction $V = V_0 xy^2$ where V_0 is a constant. Calculate the shift in energy of the ground and first excited levels of the oscillator in both first and second order in the interaction, and state how much of the degeneracy is lifted in each order.

Hint: you might find it useful to set $x = (a_x^\dagger + a_x)(\hbar/2m\omega)^{1/2}$, $y = (a_y^\dagger + a_y)(\hbar/2m\omega)^{1/2}$.

4. A quantum particle is confined to the one-dimensional interval $[-a, a]$. It is described by the time-dependent wave function

$$\psi(x,t) = \lambda \cos(\pi x/2a) \exp(-i\hbar\pi^2 t/8ma^2) - \lambda \sin(\pi x/a) \exp(-i\hbar\pi^2 t/2ma^2)$$

where λ is a constant.

(**a**) Determine the value of λ that normalizes the wave function to unity at $t = 0$,

(**b**) Check the equation of continuity involving the probability current and density, and use this fact to argue that the value of λ as derived in part (**a**) is the same for all times.

(**c**) Find the probability

$$P_{\text{left}}(t) = \int_{-a}^{0} |\psi(x,t)|^2 dx$$

for the particle to be in the left half of the interval. Compare the probabilities $P_{\text{left}}(t = 0)$ and $P_{\text{left}}(t = \tau)$ where $\tau = 4ma^2/\hbar\pi$ and explain why they are different. Interpret the time dependence of $P_{\text{left}}(t)$ in terms of a description of the physical motion of the particle.

5. (**a**) For a system with a density matrix ρ one can define the average of an operator Ω as $\Omega_{\text{av}} = \langle\Omega\rangle = \text{Tr}\rho\Omega$. In what way is the time derivative of Ω_{av} related to the Hamiltonian H of the system. Allow for the possibility that Ω could have an intrinsic dependence on time.

(**b**) A single electron with magnetic moment $\boldsymbol{\mu} = \gamma \mathbf{S}$ (where γ is its gyromagnetic ratio) is prepared with a spin projection of $\hbar/2$ in the positive x-direction. It is placed in a uniform, time-independent magnetic field $\mathbf{B}$ pointing in the z-direction to which it couples with Hamiltonian $H = -\boldsymbol{\mu} \cdot \mathbf{B}$. Determine the two-component column vector spin wave function of the electron at time t. (Take the spatial part of the wave function to be in a zero-momentum eigenstate.)

(**c**) A beam containing N electrons (each with gyromagnetic ratio γ) is prepared at time $t = 0$ in a partially polarized state in which N_+ electrons have their spin projection in the positive x-direction (i.e. spin up) and N_- have their spin projection in the negative x-direction (i.e. spin down). What is the polarization vector (the average value $\langle\boldsymbol{\sigma}\rangle$ of the Pauli matrix spin operator $\boldsymbol{\sigma}$) of the beam at time $t = 0$? The beam is now placed in a time independent external magnetic field of magnitude B pointing in the z-direction. Determine the polarization vector of the beam at some later time t.