Preliminary Exam: Quantum Mechanics, Friday January 15, 2016. 9:00-1:00

Answer a total of any **FOUR** out of the five questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than four problems, only the first four problems as listed on the exam will be graded.

Some possibly useful information

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla\psi = \mathbf{e}_x\frac{\partial\psi}{\partial x} + \mathbf{e}_y\frac{\partial\psi}{\partial y} + \mathbf{e}_z\frac{\partial\psi}{\partial z} = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} = \mathbf{e}_\rho\frac{\partial\psi}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z\frac{\partial\psi}{\partial z}.$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \;,\; H_0(x)=1 \;,\; H_1(x)=2x \;,\; H_2(x)=4x^2-2$$

$$\text{Laguerre} = L_n(r) = e^r\frac{d^n}{dr^n}\left(r^n e^{-r}\right) \;,\; \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q\frac{d^q}{dr^q}L_{n+q}(r)\;.$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \;,\; P_0(x)=1 \;,\; P_1(x)=x \;,\; P_2(x)=\frac{1}{2}(3x^2-1)\;,$$

$$\int_{-1}^{+1} dw P_\ell(w)P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} = P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} = Y_l^m(\theta,\,\phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}\;,$$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \;,\; Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \;,\; Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta-1) \;,\; Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \;,\; Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels}: \quad j_l(r) = (-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right) \;,\; n_l(r) = (-1)^{(\ell+1)}r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right)\;,$$

$$\text{with asymptotic behavior} \quad j_\ell(r) \to \frac{\cos(r-\ell\pi/2-\pi/2)}{r} \;,\; n_\ell(r) \to \frac{\sin(r-\ell\pi/2-\pi/2)}{r}\;.$$

$$j_0(r) = \frac{\sin r}{r} \;,\; n_0(r) = -\frac{\cos r}{r} \;,\; j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} \;,\; n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r}\;,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2} \;,\; n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2}\;.$$

Problem 1

(a) For the harmonic oscillator in one-dimension, write down the Hamiltonian in terms of raising ($a^\dagger$) and lowering (a) operators. What is the physical origin of the "zero point" energy? Evaluate the matrix element $< n|x^2|n >$ and relate it to E_n, the energy of the n^{th} eigenstate $|n\rangle$ of the oscillator Hamiltonian. (Here x refers to the position operator.)

(b) For a <u>non-Hermitian</u> operator A, are eigenstates corresponding to <u>different</u> eigenvalues necessarily orthogonal?

Use the eigenstates $|\alpha\rangle$ of the <u>lowering operator</u> a, given by a Poisson-weighted sum of the eigenstates $|n\rangle$ of energy of the harmonic oscillator, to examine the above question where

$$|\alpha\rangle = \exp(-|\alpha|^2/2) \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \, .$$

(Hint: First verify that the operator a is non-Hermitian and that the states $|\alpha\rangle$ are its eigenstates. Then evaluate $\langle \alpha' |\alpha\rangle$ for $\alpha \neq \alpha'$.)

Problem 2

Consider two spin 1/2 particles with spin operators $\mathbf{S}_1$ and $\mathbf{S}_2$.

(a) Write down possible (singlet and triplet) eigenstates of the total spin,

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2,$$

using $|++\rangle, |+-\rangle, |-+\rangle$, and $|--\rangle$, where $+$ and $-$ refer to up and down spin projections along a chosen z-axis. Verify that they yield the eigenvalues you expect for $\mathbf{S}^2$ and $S_z = S_{1z} + S_{2z}$.

(b) Find the eigenvalues of the operator $H = \lambda(\mathbf{S}_1 \cdot \mathbf{S}_2)$ where λ is a constant.

(c) If the interaction in part (b) is turned on at time $t = 0$ when the two electron system is in the state $|+-\rangle$, find the probabilities of finding the system in $|++\rangle, |+-\rangle, |-+\rangle$ and $|--\rangle$ at time $t > 0$.

Problem 3

(a) Consider an incident wave with momentum $\mathbf{k}$ scattering off a potential $V(\mathbf{r})$ to produce an outgoing wave with momentum $\mathbf{k}'$. Take $V(\mathbf{r})$ to vanish as $r \to \infty$. In an elastic scattering where $k' = k$ the outgoing wave function behaves as

$$\psi(\mathbf{r}) \to e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{e^{ikr}}{r} f(\mathbf{k}, \mathbf{k}')$$

where $f(\mathbf{k}, \mathbf{k}')$ is the scattering amplitude. Derive the first Born approximation for $f(\mathbf{k}, \mathbf{k}')$. Note: the Green's function that obeys $(\nabla^2 + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$ is

$$G(\mathbf{x}, \mathbf{y}) = -\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x} - \mathbf{y}|}.$$

(b) Suppose the scattering potential is radially symmetric: $V = V(r)$. Show that the corresponding first Born approximation for the elastic scattering amplitude is

$$f_{\text{Born}}(\theta) = -\frac{2m}{\hbar\kappa} \int_0^\infty dr\, r\, V(r) \sin(\kappa r),$$

where θ is the angle between the scattered wave vector $\mathbf{k}'$ and the incident wave vector $\mathbf{k}$, and $\kappa = |\mathbf{k}' - \mathbf{k}|$. Express κ in terms of θ and the magnitude k of $\mathbf{k}$. Note that in an elastic scattering k is also the magnitude of $\mathbf{k}'$.

(c) Consider the screened Coulomb potential

$$V(r) = -\frac{\lambda e^{-r/a}}{r},$$

where a is the screening radius and λ is a constant. Compute $f_{\text{Born}}(\theta)$ for this potential, and compute the first Born approximation for the total scattering cross section $\sigma_{\text{Born}}(k)$. If we let a go to infinity while holding k fixed, determine what happens to $\sigma_{\text{Born}}(k)$. Justify the answer you get on physical grounds.

Hint: the differential cross section per unit solid angle is given by $d\sigma(\theta, \phi)/d\Omega = |f(\theta, \phi)|^2$, and the total cross-section is the integral of $d\sigma(\theta, \phi)/d\Omega$ over all angles.

Problem 4

A particle with electric charge q and mass m moves in both a uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$ pointing along the z direction and a central harmonic field with a constant frequency ω. In terms of the vector potential $\mathbf{A}$ associated with $\mathbf{B}$ the Hamiltonian is given by

$$H(\mathbf{A}, \omega) = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + \frac{m\omega^2 r^2}{2}.$$

In a cylindrical coordinate system the vector potential is given by $\mathbf{A} = (A_\rho, A_z, A_\phi)$, where $A_\rho = 0,\ A_z = 0,\ A_\phi = B\rho/2$.

(a) In the case where $\omega = 0$, write down the Schrodinger equation for the wave function $\psi(z, \rho, \phi)$, and separate it into equations for the cylindrically radial (ρ), z-directional, and azimuthal (ϕ) components, and determine the eigenvalues for each separated component. Determine the eigenvalues of $H(\mathbf{A}, 0)$.

(b) Determine the change in the ground state energy of $H(0, \omega)$ to lowest order in $\mathbf{B}$.

(c) Determine the energy eigenvalues of the full Hamiltonian $H(\mathbf{A}, \omega)$.

Problem 5

A ground-state hydrogen atom is at rest in the Laboratory frame. The nucleus of this atom receives a velocity $\mathbf{V}$ in a sudden collision with a fast neutron at the time $t = 0$. The duration of this collision can be approximated by a delta-function in time and the nuclear mass can be considered to be infinite. Under these conditions, the probabilities $W_{i,f}$ for electron transitions into the new stationary states are determined by the coefficients of the expansion of the initial Laboratory frame wave function $\Psi_i(\mathbf{r}, t)$ into the electron eigenfunctions $\Psi_f(\mathbf{r}', t)$ in the Center of Mass frame:

$$W_{i,f} = |\langle \Psi_i(\mathbf{r}, t = 0)|\Psi_f(\mathbf{r}', t = 0)\rangle|^2$$

(a) Derive the formula for Galilean transformation of the electron wave function from the Laboratory frame to the Center of Mass frame :

$$\Psi'(\mathbf{r}', t) = \Psi(\mathbf{r}, t) \exp\left[-\frac{i}{\hbar}\left(m_e \mathbf{V} \cdot \mathbf{r} - \frac{m_e \mathbf{V}^2 t}{2}\right)\right]$$

where $\mathbf{r} = \mathbf{r}' + \mathbf{V}t$ and m_e is the electron mass. Hint: find the transformation for plane waves and generalize it to any electron wave function.

(b) Calculate, using the equation derived in (a), the excitation probability $W_{1s,2s}$ when an atomic electron in the $1s$ ground state is excited by the sudden collision into a $2s$ atomic electron state with electronic wave function

$$\Psi_{2s}(r, \theta, \phi) = \frac{1}{(2a_0^3)^{1/2}}\left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0} Y_{00}(\theta, \phi),$$

where $Y_{\ell m}(\theta, \phi)$ are the spherical harmonics and a_0 is the Bohr radius. The axis of quantization of the angular momentum of the electron is along the direction of $\mathbf{V}$.

(c) Calculate the total probability of atomic ionization and excitation.