Preliminary Examination: Quantum and Statistical Mechanics, 1/12/2007

Answer a total of **SIX** questions, choosing **TWO** from section A and **FOUR** from section B. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book, or on consecutively numbered sheets of paper stapled together. Make sure you clearly indicate who you are, and the problem you are answering. Double-check that you include everything you want graded, and nothing else.

Possibly Useful Information

$$N! \approx N \ln N - N \quad \text{as} \quad N \to \infty$$

$$\int_{-\infty}^{+\infty} dx\ \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp(\frac{\beta^2}{4\alpha}) \text{ with } \mathrm{Re}(\alpha) > 0, \quad \int_0^{\infty} dx\, x\ \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

First few spherical harmonics

$$Y_{00} = (\frac{1}{4\pi})^{1/2}, \quad Y_{10} = (\frac{3}{4\pi})^{1/2} \cos\theta, \quad Y_{1\pm1} = \mp(\frac{3}{8\pi})^{1/2} \sin\theta\ \exp(\pm i\phi)$$

$$Y_{20} = (\frac{5}{16\pi})^{1/2}\ (3\cos^2\theta - 1), \quad Y_{2\pm1} = \mp(\frac{15}{8\pi})^{1/2}\ \sin\theta\cos\theta\ \exp(\pm i\phi),$$

$$Y_{2\pm2} = (\frac{15}{32\pi})^{1/2}\ \sin^2\theta\ \exp(\pm 2i\phi)$$

Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

Section A

A1. (a) Consider a simple chemical reaction $A + B \leftrightarrow AB$. As the double arrow indicates, all chemical reactions go both ways. Show that in a mixture of reactants the condition for chemical equilibrium is expressed in terms of chemical potentials as $\mu_A + \mu_B = \mu_{AB}$.

(b) Show that in an ideal gas where the only relevant degree of freedom is the center-of-mass motion of the atoms or molecules, the chemical potential in the classical (high temperature/low density) limit is $\mu = kT \ln(n\lambda^3)$, where $\lambda = (2\pi\hbar^2/mkT)^{1/2}$ is the usual thermal de Broglie wavelength.

(c) Consider the formation of diatomic molecules A_2 out of atoms A. Assume that the binding energy of the molecule is I, so that the difference in internal energy between a molecule and a pair of atoms is $-I$. Show that in the limit when the atoms and molecules may be regarded as classical ideal gases, except for the formation of the molecules, the equilibrium densities of the atoms and molecules satisfy $n_{A_2}/n_A^2 = \sqrt{8}\lambda_A^3 \exp(I/kT)$. The chemical-equilibrium densities of atoms and molecules depend manifestly on the de Broglie wavelength, i.e., on quantum mechanics, even at room temperature.

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A2. A system has 3 single-particle energy levels, ϵ_0, ϵ_1 and ϵ_2. Suppose the system contains two noninteracting identical particles, and each level may accommodate 0, 1 or 2 particles. Assume that the spacing between the levels is Δ (i.e., $\epsilon_2 = \epsilon_1 + \Delta, \epsilon_1 = \epsilon_0 + \Delta$, with $\Delta > 0$). *Hint: Do not blindly use Fermi-Dirac or Bose-Einstein statistics.*

(a) How many different 'two-particle' states can exist here? Are they degenerate?

(b) Show that at any temperature T, the mean occupancy $< n_1 >$ of the single-particle level having energy ϵ_1 is
$< n_1 >= (x + 2\ x^2 + x^3)/(1 + x + 2\ x^2 + x^3 + x^4)$ and identify x.

(c) Find the mean energy $< E >$ of the two-particle system at temperature T. Evaluate the limiting value of $< E >$ as T goes to zero. Does this make sense? Why?

A3. Assume you have an ideal Fermi gas in a volume V, consisting of N noninteracting, neutral, spin-1/2 particles with magnetic moment μ and mass m at zero temperature. This gas is placed in a uniform magnetic field $\mathbf{B}$ (choose $+z$ direction to be the field direction, i.e., $\mathbf{B} = B\ \hat{\mathbf{z}}$).

(a) Sketch the problem, i.e., plot the single particle energies for each spin as a function of momentum. Where is the Fermi energy, E_F, in this plot?

(b) Write down expressions for the particle numbers N_+ and N_- for the two types of spins as integrals over momenta and hence show that,

$$N_+^{\frac{2}{3}} - N_-^{\frac{2}{3}} = \frac{4m\mu B}{\hbar^2}\left(\frac{V}{6\pi^2}\right)^{\frac{2}{3}}.$$

Hint: Remember that

$$\sum_k \to \frac{V}{(2\pi)^3}\int d^3k.$$

(c) Now assume that the magnetic field $\mathbf{B}$ is weak and therefore the difference, $2n \equiv N_+ - N_- \ll N$. Calculate n and the resulting magnetization, $M \equiv \mu(N_+ - N_-)$, of the gas approximately.

(d) What do you understand by saturated magnetization? Using the result from part (b), find the magnitude of the threshold magnetic field at which the magnetization saturates.

Section B

B1. Consider a two-level system (states $|1\rangle$ and $|2\rangle$) coupled to an external time dependent driving field, so that the Hamiltonian reads

$$\frac{H}{\hbar} = \omega_1 |1\rangle\langle 1| + \omega_2 |2\rangle\langle 2| + f(t)(|2\rangle\langle 1| + |1\rangle\langle 2|)\,.$$

Take the driving field to be of the form

$$f(t) = \frac{\lambda}{\sqrt{\pi}\tau}\left\{\exp\left[-\left(\frac{t+\frac{T}{2}}{\tau}\right)^2\right] + \exp\left[-\left(\frac{t-\frac{T}{2}}{\tau}\right)^2\right]\right\}.$$

This represents two pulses of length τ hitting the system at times $\mp T/2$. Let us assume that the amplitude of the driving field $\propto \lambda$ is "very small." Moreover, let us take the pulse width τ to be the smallest time parameter in the problem, so that the excitation pulses are, in effect, delta function kicks. Before the pulses, the system starts in the state $|1\rangle$. Show that the probability that the system is in the state $|2\rangle$ after the two pulses approximately equals

$$P_2 \simeq 2\lambda^2\{1 + \cos[(\omega_2 - \omega_1)T]\}\,.$$

This is the basic idea of what is known as Ramsey fringes: By making the separation of the two pulses T longer, one may measure the energy difference between the states with increasing accuracy.

B2. In an atom the spins of the electrons and of the nucleus are coupled by what is known as hyperfine interaction. The hyperfine interaction is invariant under a simultaneous rotation of all spins.

(a) Describe qualitatively the ensuing hyperfine structure in the ground state of hydrogen. Of course, an electron and a proton both have spin 1/2.

(b) In a Stern-Gerlach experiment an inhomogeneous magnetic field separates the atoms according to the projection of their angular momentum on the direction of the magnetic field. Describe the outcome of a Stern-Gerlach experiment (how many directions, ratios of the numbers of atoms going in each direction) for ground-state hydrogen in the usual case when the thermal energy kT greatly exceeds the energy scale of the hyperfine structure.

B3. Consider the following Hamiltonian of a system, $H = \mathbf{L}^2/(2I) - \mu\ L_z$, where $\mathbf{L}$ and L_z denote orbital angular momentum operator and its z component respectively. (I and μ are constants.) Assume that at time $t = 0$, the system is found in the state

$$\psi(\theta, \phi, 0) = c_1\ \sin\theta\cos\phi + c_2\ \cos^2\theta,$$

where c_1 and c_2 are constants and θ and ϕ are the polar and azimuthal angles. *Hint: Some knowledge of spherical harmonics may be useful.*

(a) What are the possible values of $\mathbf{L}^2$ and L_z that a measurement will find? What are their respective probabilities?

(b) Evaluate the fluctuations in energy, $< \psi|H^2|\psi > - < \psi|H|\psi >^2$, and explain why this expression is not necessarily zero.

(c) How does ψ evolve in time? (i.e., write down an expression for $\psi(\theta, \phi, t)$ at time t.) Do you expect the probabilities in (a) and fluctuations in (b) to vary with time? Why?

B4. The Hamiltonian for an electron in a one dimensional harmonic oscillator potential and a time independent electric field, $\mathcal{E}$, is given by the quantum mechanical Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 - e\mathcal{E}x,$$

with e being the charge of an electron. The other symbols have their usual meanings.

(a) Using the Heisenberg representation, find the equations that determine the time derivatives of x and p.

(b) Using (a) or otherwise, obtain complete solutions for $x(t)$ and $p(t)$ in terms of their $t = 0$ values, i.e., $x(0)$ and $p(0)$. Is there a classical analog here? Briefly explain.

(c) Evaluate the commutator $[x(t_1), x(t_2)]$, where t_1 and t_2 are not necessarily equal, and show that it need not vanish for arbitrary t_1 and t_2. Is there a classical analog here? Briefly explain.

B5. (a) Assume a one-dimensional (1D) potential with

$$U(x) = \begin{cases} \infty & \text{for} \quad x < 0 \\ -U_0 & \text{for} \quad 0 \leq x \leq a \\ 0 & \text{for} \quad x > a \end{cases}$$

For which minimum value of U_0 (> 0) does this system support at least one bound state?

(b) Now assume a spherical well potential in 3D with

$$V(r) = \begin{cases} -V_0 & \text{for} \quad r \leq a \\ 0 & \text{for} \quad r > a \end{cases}$$

Assume a spherically symmetric ($\ell = 0$) wavefunction ψ. Show that this problem can be reduced to the one-dimensional problem in part (a). To do this, make an ansatz $\psi(r, \theta, \phi) = r^s R(r)$. For which s does $R(r)$ solve the 1D problem? What, therefore, is the minimum V_0 (> 0) that can support at least one bound state? *Hint: The Laplace operator in polar coordinates can be written as* $\nabla^2 = \frac{1}{r^2}\partial_r r^2 \partial_r - \frac{\vec{L}^2}{\hbar^2 r^2}$.

(c) Show that this is possible in general, if we compare $U(x)$ in 1D and a spherically symmetric $V(r)$ in 3D:

$$U(x) = \begin{cases} \infty & \text{for} \quad x < 0 \\ -U_0(x) & \text{for} \quad 0 \leq x \leq a \\ 0 & \text{for} \quad x > a \end{cases} \qquad \text{and} \qquad V(r) = \begin{cases} -V_0(r) & \text{for} \quad r \leq a \\ 0 & \text{for} \quad r > a \end{cases}$$

Note the boundary conditions!

B6. Assume a two-dimensional harmonic oscillator,

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2),$$

where the symbols have their usual meanings.

(a) Use perturbation theory to find out what happens to the first excited state if one adds a potential $V = \alpha xy$, with $\alpha \ll m\omega^2$. (*Hint: This problem might be easier to solve using raising and lowering operators.*)

(b) Compare with the exact solution, using the fact that

$$\frac{k}{2}(x^2 + y^2) + \alpha xy = \frac{k+\alpha}{4}(x+y)^2 + \frac{k-\alpha}{4}(x-y)^2.$$