Preliminary Exam: Quantum Physics 1/19/2004, 9:00-3.00

Answer a total of **SIX** questions of which at least **TWO** are from section **1**, and at least **THREE** are from section **2**. Put each of your solutions in a separate answer book.
Some possibly useful information:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\int_0^\infty dx\, e^{-a^2x^2} = \frac{\pi^{1/2}}{2a}, \quad \int_0^\infty dx\, xe^{-a^2x^2} = \frac{1}{2a^2}, \quad \int_0^\infty \frac{zdx}{(e^{ax}-z)} = -\frac{\ln(1-z)}{a}$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$$

$$\text{associated Laguerre} = L_{n+l}^{2l+1}(r) = \sum_{k=0}^{n-l-1}(-1)^{k+2l+1}\frac{[(n+l)!]^2 r^k}{(n-l-1-k)!(2l+1+k)!k!}$$

$$\text{Legendre polynomial} = P_l(w) = \frac{1}{2^l l!}\frac{d^l}{dw^l}(w^2-1)^l$$

$$\text{associated Legendre polynomial} = P_l^m(w) = (1-w^2)^{|m|/2}\frac{d^{|m|}}{dw^{|m|}}P_l(w)$$

$$\text{spherical harmonic} = Y_l^m(\theta,\,\phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2}P_l^m(\cos\theta)e^{im\phi}$$

$$\text{spherical Bessels}: \quad j_l(r) = R_l(r)\frac{\sin r}{r} + S_l(r)\frac{\cos r}{r}, \qquad n_l(r) = R_l(r)\frac{\cos r}{r} - S_l(r)\frac{\sin r}{r},$$

$$\text{where} \quad R_l(r) + iS_l(r) = \sum_{s=0}^{l}\frac{i^{s-l}(l+s)!}{2^s s!(l-s)!}r^{-s},$$

$$\text{and with asymptotic behavior} \quad j_\ell(r) \to \frac{\sin(r-\ell\pi/2)}{r}, \quad n_\ell(r) \to \frac{\cos(r-\ell\pi/2)}{r}.$$

Section 1: Statistical Mechanics

1.1 Consider a quantum system of N independent particles that "live" in a volume V with energy levels $\epsilon = m^{1+\nu}\Delta/V$, where $m = 0, 1, 2, ...$, and $-1 \le \nu \le 1$.
(a) What is the density of states $a(\epsilon)$ for this system, if Δ is small enough to assume a quasi-continuous spectrum?
(b) Show that in the case of $\nu = 0$, the density of particles in the excited states ($m \neq 0$) is given by

$$\frac{N_e}{V} = -\frac{\ln(1-z)}{\beta\Delta}$$

where $\beta = 1/kT$ is the inverse temperature and $z = e^{\beta\mu}$ is the fugacity of the system.
(c) Show that for $\nu = 0$ this system does not exhibit a Bose-Einstein condensation at any finite temperature.

1.2 In ferromagnets there exist quantized waves of magnetization called spin waves. The dispersion relation for these types of waves is $\omega = ck^2$, where c is a material constant.
(a) Show that the chemical potential, μ, of the quanta of spin waves (we might call them "Spinons") is zero.
(b) What is the spin-wave density of states $a(\omega)$ in a three-dimensional ferromagnet?
(c) Find a power law for the low-temperature heat capacity due to spin waves of the form $C_V = AT^\nu$ (i.e. determine the parameters A and ν in terms of c, the volume of the ferromagnet V, and the number of spin sites N).

1.3 Consider a system of classical hard spherical particles, each of volume $b/8$, enclosed in a volume V.
(a) What is the canonical partition function $Q_N(V,T)$ for this system? (Hint: Assume that the volume accessible to the $(k+1)$st particle is given by $V-kb$).
(b) Derive the equation of state of this system in the form of a virial expansion, retaining only the first and the second term. (Hint: $\ln(1-x) = -x + O(x^2)$.)
(c) Briefly discuss how the assumptions made in parts (a) and (b) affect the validity of the results.

Section 2: Quantum Mechanics

2.1 (a) A single electron is prepared with its spin quantized in the positive y direction. The state is then rotated through an angle θ about the x-axis.
(i) What is the wave function of the rotated state?
(ii) Of what operator is the rotated state an eigenvector, and what is the associated eigenvalue?

(b) A beam of N electrons is prepared so that M of them have their spins quantized with spin up in the y-direction with the rest of them being quantized with their spins down in the y-direction. What is the density matrix for this system? What are the expectation values of the x, y and z components of the total spin operator for this beam?

2.2 (a) Consider the vector potential $\bar{A} = (-yB, 0, 0)$ where B is a pure constant. What magnetic field does $\bar{A}$ represent?

(b) An electron is placed in this magnetic field. Write down both the Schrodinger and Dirac equations for this system.

(c) Make the non-relativistic reduction of this Dirac equation to obtain the Schrodinger equation and its first relativistic correction. In particular determine the g factor of the electron.

A convenient representation of the Dirac matrices is given as:

$$\alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \quad \alpha_y = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}, \quad \alpha_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

2.3 (a) A quantum-mechanical system has a time-independent Hamiltonian H_0 and an eigenspectrum of states $|n\rangle$ with energies E_n. While in its ground state it is subjected to a time-dependent perturbation $V(t)$ starting at a time $t=0$. Derive the first order probability for finding this system in any other of its states at a later time t.

(b) Suppose the above system is a rigid rotator of moment of inertia I, Hamiltonian $H_0 = \bar{L}^2/2I$ and electric dipole moment $\bar{d}$, and that it is subjected to a perturbation

$$V(t) = -\bar{d} \cdot \bar{E}(t)$$

due to a time-dependent external electric field

$$\bar{E}(t) = \hat{z} E_0 e^{-t/\tau}$$

which points in the z-direction and which is switched on at time $t=0$. Determine to which of its excited states the rotator can make transitions in lowest order in $V(t)$, and calculate the transition probabilities for finding the rotator each of these states at time $t=\infty$.

(c) If instead of being in its ground state, the rotator was in its first excited state at the time $V(t)$ was switched on, determine the states to which it would be able to make transitions. (For this part do not calculate the transition probabilities.)

2.4 Consider the quantum mechanical creation and annihilation operators $a^\dagger$ and a, which satisfy the fundamental commutation relations

$$[a,\, a^\dagger] = \hbar \mathbf{1}$$

(a) Show that the "coherent state"

$$|\alpha\rangle = e^{-\hbar|\alpha|^2/4}\, e^{\alpha a^\dagger}\, |0\rangle$$

is an eigenstate of the annihilation operator a.

(b) Expand the coherent state $|\alpha\rangle$ in terms of normalized eigenstates of the number operator, and hence determine the probability for the coherent state $|\alpha\rangle$ to contain n quanta.

(c) Show that the coherent state $|\alpha\rangle$ is a "minimum uncertainty state", in the sense that it saturates the uncertainty principle bound for position and momentum uncertainties, where $x = (a + a^\dagger)/\sqrt{2}$ and $p = -i(a - a^\dagger)/\sqrt{2}$.

2.5 Consider a nonrelativistic charged particle in the presence of an external classical electromagnetic field with scalar potential Φ and vector potential $\vec{A}$.

(a) Write down the Schrödinger equation for this system, and show that the conserved probability current density is

$$\vec{J} = -\frac{i\hbar}{2m}\left[\psi^* \left(\vec{\nabla} - \frac{ie}{\hbar c}\vec{A}\right)\psi - \left(\left(\vec{\nabla} - \frac{ie}{\hbar c}\vec{A}\right)\psi\right)^* \psi\right]$$

(b) Suppose there is just an external magnetic field, of constant strength B. Choose the gauge $\vec{A} = B(0, x, 0)$ and show that the solutions can be written as

$$\psi(x, y, z) = \frac{1}{2\pi} e^{ik_y\, y}\, e^{ik_z\, z}\, \phi_n\left(x - \frac{\hbar c k_y}{eB}\right)$$

where ϕ_n are the eigenstates of a one-dimensional harmonic oscillator system with frequency $\omega_c = eB/mc$.

(c) Describe the energy spectrum of the system in part (b).

2.6 (a) Derive the first Born approximation for the elastic scattering amplitude, stating clearly the relevant assumptions. Note that the Helmholtz Green's function $G(\vec{x}, \vec{y})$ satisfying $(\nabla^2 + k^2)G(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y})$ is

$$G(\vec{x}, \vec{y}) = -\frac{e^{ik|\vec{x}-\vec{y}|}}{4\pi|\vec{x} - \vec{y}|}$$

(b) Suppose the scattering potential is radially symmetric: $V = V(r)$. Show that the corresponding first Born approximation for the scattering amplitude is

$$f_{\rm Born}(\theta) = -\frac{2m}{\hbar\kappa}\int_0^\infty dr\, r\, V(r)\, \sin(\kappa r)$$

where θ is the angle between the scattered wave vector $\vec{k}'$ and the incident wave vector $\vec{k}$, and $\kappa = |\vec{k}' - \vec{k}|$.

(c) Consider "soft-sphere" scattering for which the potential is

$$V(r) = V_0 \quad , \quad r < a \ ,$$

$$V(r) = 0 \quad , \quad r > a \ .$$

Compute the first Born approximation for the *cross section*. What is the leading low energy behavior of this cross section?