Preliminary Exam : Quantum Physics

January 21, 2003, 9:00 a.m. - 1:00 p.m.

Please answer **3 QUESTIONS** from each of the two sections.
Please use a separate book **FOR EACH QUESTION**.

Section I: Statistical Mechanics

1. Consider a degenerate Fermi free electron gas consisting of N electrons in a volume V.

   (a) Show that the density of electronic states is given by

   $$D(\epsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}$$

   and find the Fermi energy at $T = 0$: $\epsilon_F(0)$.

   (b) Show that in the ground state this Fermi gas exerts a pressure given by

   $$p = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{5/3}.$$

   (c) Find an expression for the entropy of this Fermi gas in the region where $k_BT \ll \epsilon_F(0)$, and the electronic heat capacity can be written as $C_{el} = \pi^2 N k_B T/(2\epsilon_F(0))$.

2. Consider a system of N non-interacting, extreme relativistic particles. The contribution of each particle to the total energy is given by $\epsilon_i = |\mathbf{p}_i|c$, where c denotes the speed of light.

   (a) Show that the canonical partition function $Q_N(T, V)$ is given by

   $$Q_N(T,V) = \frac{V^N}{N!h^{3N}}8\pi\left(\frac{kT}{c}\right)^{3N}.$$

   Calculate the internal energy U and heat capacity $C_V = (\partial U/\partial T)_V$ of this system.

   (b) What is the equation of state of this system?

   (c) What is the isobaric heat capacity $C_P = (\partial H/\partial T)_P$, where H is the enthalpy?

3. Consider an idealized model of a white dwarf star as a system of N relativistic electrons in its ground state, held together by the gravitational effect of $N/2$ motionless helium nuclei. The electron gas is essentially an ideal Fermi gas in its ground state, whose repulsive zero-point pressure is balanced by the gravitational pressure of the helium nuclei.

   (a) Express the Fermi momentum for the electron gas system in terms of the number density N/V.

   (b) Treating the electrons relativistically, with single particle energies given by the expression $\epsilon = \sqrt{(pc)^2 + (mc^2)^2}$, find a simple integral expression for the zero temperature ground state energy E_0 of the Fermi gas.

   (c) Consider the extreme relativistic limit where $p_F \gg mc$. Given the following integral expression

$$\int_0^{x_F} dx\, x^2 \sqrt{1+x^2} \sim \frac{1}{4}\, x_F^4 + O(x_F^2) \quad , \quad x_F \gg 1$$

   compare with your answer in (b) to find the leading behavior of E_0 in the extreme relativistic limit. Hence show that the pressure P is proportional to $\hbar c\,(N/V)^{4/3}$ in this limit.

   (d) The Chandrasekhar mass M_c corresponds to the situation where the repulsive Fermi pressure balances the attractive gravitational pressure. Given the gravitational pressure

$$P_{\text{grav}} = -\alpha \frac{GM^2}{R^4}$$

   where α is some numerical constant and G is the gravitational constant, use the result of part (c) to argue on dimensional grounds that M_c can be expressed in terms of the constants c, $\hbar$, G, and m_p, where G is the gravitational constant and m_p is the proton mass.

4. The isothermal-isobaric ensemble, the system is considered to be in contact with a bath of enthalpy $H = U + PV$.

   (a) Assuming the states accessible to the system have energies E_r and volumes V_r, what is the isothermal-isobaric partition function $\Delta_N(T, P)$?

   (b) Show that the mean square fluctuations of the enthalpy $\langle H^2 \rangle - \langle H \rangle^2$ are proportional to the isobaric heat capacity $C_P = \left(\frac{\partial H}{\partial T}\right)_{N,P}$.

   (c) Calculate the mean square fluctuations of the energy U, and express the result in terms of the isochoric heat capacity C_V and the volume expansion coefficient α_P.

Section II: Quantum Mechanics

5. Let J_1 and J_2 be the respective angular momentum of the individual particles of a two-particle system. The combined system has total angular momentum $J = J_1 + J_2$.

   (a) Show that

   $$J^2 = J_1^2 + J_2^2 + 2J_1^z J_2^z + (J_1^+ J_2^- + J_1^- J_2^+),$$

   where $J^{\pm} = J^x \pm iJ^y$ are the raising/lowering operators.

   (b) Now, consider a system consisting of only two electron spins, each one described by spin up (α) and spin down (β). Express the possible eigenstates of total angular momentum in terms of product wavefunctions for the individual spins.

   (c) Using the result from part (a), show that one of your triplet states from part (b) is indeed characterized by total spin $S = 1$, and z-component of total spin $m = 1$.

Note:

$$J^{\pm}|j,m\rangle \quad = \quad \hbar\,[j(j+1) - m(m \pm 1)]^{1/2}\,|j, m \pm 1\rangle$$

6. Consider a hydrogen atom placed in a weak uniform external electric field of magitude E_{ext}. To first order in the external field, determine the effect of this external field on the energies of the $n = 1$ and $n = 2$ states of hydrogen.

Note: the relevant normalized spherical harmonics and hydrogen radial wavefunctions are :

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} \quad , \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \quad , \quad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

$$R_{10} = \frac{2}{a^{3/2}}e^{-r/a} \quad , \quad R_{20} = \frac{1}{\sqrt{2}a^{3/2}}\left(1 - \frac{r}{2a}\right)e^{-r/(2a)} \quad , \quad R_{21} = \frac{1}{\sqrt{24}a^{3/2}}\frac{r}{a}\,e^{-r/(2a)}$$

and you may need the following integrals :

$$\int_0^{\pi} \sin\theta\,\cos^2\theta\,d\theta = \frac{2}{3} \quad , \quad \int_0^{\infty} dr\,r^k\,e^{-r} = k!$$

7. A spin-$\frac{1}{2}$ Dirac particle obeys the equation of motion

$$i\hbar\left(\gamma^0\frac{\partial}{\partial t}\psi+\gamma\cdot\nabla\psi\right)=m(c^2+\frac{1}{2}\omega^2r^2)\psi$$

Adopting the usual convention for the Dirac matrices

$$\gamma^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad\gamma=\begin{pmatrix}0&\sigma\\-\sigma&0\end{pmatrix}$$

gives rise to discrete solutions of the form

$$\psi(r,\theta,\phi)=\begin{pmatrix}\psi_1(r)\;\chi_1(\theta,\phi)\\\psi_2(r)\;\chi_2(\theta,\phi)\end{pmatrix}$$

where each ψ_i is a radial wave function and each χ_i is a two-component spinor.

(a) Consider solution ψ_{njm} which is a state of good total angular momentum $J^2\psi_{njm}=\hbar^2j(j+1)\psi_{njm}$, and $J_3\psi_{njm}=\hbar m\psi_{njm}$. Use this property to expand χ_1 and χ_2 in the product basis $Y_l^m(\theta,\phi)\left|m_s=\pm\frac{1}{2}\right\rangle$, expressing the coefficients of the expansion for general j,m in terms of Clebsch-Gordan symbols.

(b) Show that the parity operator $\mathcal{P}$ for this system, such that if $\psi(r)$ is a solution then so is $\mathcal{P}\psi(-r)$, is given by the Dirac matrix γ^0. What is the parity of the state ψ_{njm} found in part 7a?

(c) Without solving a differential equation, write down the energy of each eigenstate ψ_{njm} in the nonrelativistic limit, $mc^2\gg\hbar\omega$? Be careful to list both positive and negative energy solutions, indicating the parity of each.

8. An atom modeled as a two-level system with eigenstates of the free Hamiltonian given by $H_0\psi_0=\varepsilon_0\psi_0$ and $H_0\psi_1=\varepsilon_1\psi_1$ is fixed inside an optical cavity. The cavity is designed to confine a single mode of frequency ω without loss, where ω is the resonant frequency of the atomic transition. The full Hamiltonian for the atom + cavity, considered in one dimension, is

$$H=H_0+\frac{1}{2}\hbar\omega(a_+a_-+a_-a_+)+eEx$$

where the electric field E in the cavity can be written in terms of the raising and lowering operators a_+ and a_- (which increase and decrease the photon count in the cavity by one) as

$$E=E_0\left(a_+e^{i\omega t}+a_-e^{-i\omega t}\right)$$

A general eigenstate of H can be written as

$$\Psi_n=\cos\theta\;\psi_0\;|n\rangle+\sin\theta\;\psi_1\;|n-1\rangle$$

where rapidly oscillating terms have been neglected.

(a) Use the relation $[a_-,a_+]=1$ to show that $a_-|n\rangle=\sqrt{n}\,|n-1\rangle$ and $a_+|n\rangle=\sqrt{n+1}\,|n+1\rangle$ for normalized cavity states $|n\rangle$ with a definite number of photons n.

(b) Use the variational principle to derive the value of θ for which the matrix element $(\Psi_n,H\Psi_n)$ is stationary. You may denote the real constant matrix element $(\psi_1,eE_0x\,\psi_0)$ by the symbol d, which should not be assumed to be small.

(c) What is the ground state of the system?