Preliminary Examination: Electricity and Magnetism, 08/23/2018

Answer a total **THREE** questions out of **FOUR**. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book, or on consecutively numbered sheets of paper stapled together. Make sure you clearly indicate who you are and the problem you are solving. Double-check that you include everything you want graded, and nothing else.

1. An otherwise free non-relativistic charged particle having mass m and charge e moves in a uniform magnetic field $\vec{B} = B_0 \hat{e}_z$ where $\hat{e}_z$ is the unit vector along z-axis.

   a) Suppose that at $t = 0$ the particle is located at the origin and moving at velocity $\vec{v}_0 = v_0 \hat{e}_x$ where $\hat{e}_x$ is the unit vector along x-axis. Determine the particle's subsequent position $\vec{r}(t)$ and velocity $\vec{v}(t)$ as a function of time t. Describe the resulting motion in words. You may neglect the effects of radiation damping.

   b) If the initial velocity $\vec{v}_0 = v_{0x}\hat{e}_x + v_{0y}\hat{e}_y$ where $\hat{e}_x, \hat{e}_y$ are the unit vectors along x- and y-axes, find the subsequent position $\vec{r}(t)$ and describe the resulting motion.

2. Consider a very long uniform solenoid of radius R and N turns per unit length, carrying electric current I. Concentric with the solenoid are two long cylindrical shells of length ℓ, one with radius $a < R$ and the other with radius $b > R$. The inner cylinder carries total charge $+Q$ distributed evenly over its surface, while the outer cylinder carries total charge $-Q$, also uniformly distributed over its surface. You should take the limit $\ell \gg R$ and ignore end effects.

   a) If neither of the charged cylinders are rotating, what are the electric and magnetic fields everywhere in space?

   b) Under the conditions in part (a), what is the total angular momentum $\vec{L}$ carried by the electromagnetic fields? *Hint:* You may use the fact that crossed $\vec{E}$ and $\vec{B}$ fields carry linear momentum density $\vec{s} = \epsilon_0 \vec{E} \times \vec{B}$ and then compute the angular momentum as the integral of $\vec{r} \times \vec{dp}$ over the field volume.

   c) As the current in the solenoid is gradually reduced to zero, the cylinders experience a torque about their axis. Calculate the final angular momentum of each, supposing that they are free to rotate without friction. Show that the sum of their final angular momenta is equal to the initial angular momentum found in part (b). You may assume that their moments of inertia I_a and I_b are large enough that the magnetic induction arising from their rotation can be neglected relative to that due to the solenoid current.

3. An ideal (point) dipole $\vec{p} = p\,\hat{e}_x$ is held a distance $\vec{r}_0 = r_0\hat{e}_z$ from the center of a grounded conducting sphere of radius R, where r_0 is a positive constant ($r_0 > R$) and $\hat{e}_x, \hat{e}_z$ are the unit vectors along x- and z-axes. The dipole vector is perpendicular to the z-axis.

   a) Determine the parameters of the induced image dipole $\vec{p}_{im}$.

   b) Find the potential $\Phi(\vec{r})$ in the region outside the conducting sphere ($r > R$).

   c) Calculate the energy of the dipole interaction with the conducting sphere.

   Hint: The parameters of the image dipole can be obtained using the charge image value $q_{im} = -q\ R/r_0$ and location $r_{im} = R^2/r_0$ induced by the point charge q located at the distance r_0 relative to the center of grounded sphere.

4. A thick conducting slab, extending from $z = -a$ to $z = +a$, carries a non-uniform current described by the current density $\vec{j}(\vec{r}) = j_0\,\frac{z}{a}\,\hat{e}_x$, where j_0 is a positive constant and $\hat{e}_x$ is the unit vector parallel to the x-axis. The density of electric current is zero outside the slab.

   a) Show that the magnetostatic Ampère's law can be written in the form of a Poisson's equation $\nabla^2\vec{A}(\vec{r}) = -\mu_0\vec{j}(\vec{r})$ for the vector potential $\vec{A}$ induced by the electric current in the two regions $|z| < a$ and $|z| \geq a$. Justify, that the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ is zero outside the current slab $|z| \geq a$ because of the specific symmetry of $\vec{j}(\vec{r})$.

   b) Find the vector potential $\vec{A}$ in the entire space by solving Poisson's equation from part (a) with the boundary conditions $\vec{A}(\vec{r})|_{z=0} = 0$ and $\vec{B}(\vec{r})|_{z=0} = \frac{1}{2}\mu_0 j_0 a\,\vec{e}_y$, where $\vec{e}_y$ is the unit vector of the y-axis.

   c) Calculate the force $\vec{F}$ exerted on an ideal magnetic dipole $\vec{m} = m_0(\vec{e}_y + \vec{e}_z)/\sqrt{2}$, where m_0 is a positive constant, and $\vec{e}_y$ and $\vec{e}_z$ are the unit vectors of the y- and z-axes. The location of the magnetic moment can be inside or outside the slab.

   Hint: $\vec{A}(\vec{r})$ and $\vec{B}(\vec{r})$ are continuous functions of the coordinates because there are no surface electric currents in this problem.

Vector Formulas

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$
$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$
$$\nabla \times \nabla \psi = 0$$
$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$
$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$
$$\nabla \cdot (\psi \mathbf{a}) = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}$$
$$\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a}$$
$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$
$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$
$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla \cdot \mathbf{x} = 3 \qquad \nabla \times \mathbf{x} = 0$$
$$\nabla \cdot [\mathbf{n} f(r)] = \frac{2}{r} f + \frac{\partial f}{\partial r} \qquad \nabla \times [\mathbf{n} f(r)] = 0$$
$$(\mathbf{a} \cdot \nabla)\mathbf{n} f(r) = \frac{f(r)}{r} [\mathbf{a} - \mathbf{n}(\mathbf{a} \cdot \mathbf{n})] + \mathbf{n}(\mathbf{a} \cdot \mathbf{n}) \frac{\partial f}{\partial r}$$
$$\nabla(\mathbf{x} \cdot \mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla \cdot \mathbf{a}) + i(\mathbf{L} \times \mathbf{a})$$

where $\mathbf{L} = \frac{1}{i} (\mathbf{x} \times \nabla)$ is the angular-momentum operator.

Theorems from Vector Calculus

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\, da \qquad \text{(Divergence theorem)}$$

$$\int_V \nabla \psi\, d^3x = \int_S \psi \mathbf{n}\, da$$

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da$$

$$\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi)\, d^3x = \int_S \phi \mathbf{n} \cdot \nabla \psi\, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, d^3x = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n}\, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla \psi\, da = \oint_C \psi\, d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]$$
$$+ \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$