ELECTRICITY AND MAGNETISM

Preliminary Examination

August 25, 2011

9:00am - 12:00pm in P-121

Answer a total of **THREE** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on individual sheets of paper. Make sure you clearly indicate who you are and the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

On the last two pages you find the forms of the standard vector calculus operations in the three most common coordinate system and general formulas for the Legendre polynomials.

Problem 1. Two point charges q and $-q$ are located on the z axis of a Cartesian coordinate system at $z = a$ and $z = -a$, respectively.

(a) Find the electrostatic potential as an expansion in Legendre polynomials and powers of the radial distance r for both $r > a$ and $r < a$.

(b) Keeping the product $2qa = p$ constant, take the limit as $a \to 0$ and find the potential for $r \neq 0$.

(c) Suppose now that the system in (b) is surrounded by a grounded spherical shell of radius b concentric with the origin. Find the electrostatic potential and the electrostatic field everywhere inside the shell.

Problem 2. A two-dimensional box has conducting walls defined by four sides $x = 0$, $y = 0$, $x = a$, and $y = b$. The side $x = 0$ is held at $\frac{\partial \Phi}{\partial x} = 0$ and the side $x = a$ is held at a potential $\Phi = \Phi_0 =$ constant. The other two sides are at zero potential. Solve Laplace's equation using the method of the separation of variables, and find the potential $\Phi(x, y)$ at any point inside the box.

Problem 3. A non-uniform electric current propagates along the z axis. The current density is given in Cartesian coordinates by the formula

$$\mathbf{j} = j_0 \, e^{-|x|/\lambda} \, \mathbf{e}_z,$$

where j_0 and λ are positive constants and $\mathbf{e}_z$ is the unit vector of the z axis. Find the vector potential of the magnetic field $\mathbf{A}(x, y, z)$ everywhere by solving the Poisson equation $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}$. The reference point for the vector potential is at $\mathbf{r} = 0$: $\mathbf{A}(0) = 0$.

Problem 4. A plane disk of radius R_0 is uniformly charged to the surface charge density σ. The disk is rotating with the angular velocity ω about the symmetry axis z that is perpendicular to the disk surface.

(a) Calculate the magnetic moment of this disk $\mathbf{m}_d$.

(b) Find the magnetic field $\mathbf{B}$ and vector-potential $\mathbf{A}$ induced at large distances r ($r \gg R_0$) from the disk, considering the magnetic field induced by the magnetic moment $\mathbf{m}_d$.

(c) Find the energy of interaction between this disk and a point magnetic dipole moment $\mathbf{m}_p = m_p \mathbf{e}_z$ located at $\mathbf{r} = \mathbf{r}(r, \theta, \phi)$, with $r \gg R_0$.

Standard vector operations in three common coordinate systems

Cartesian coordinates x, y, z

$$\begin{aligned}
\nabla &= \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{e}}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{e}}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}$$

cylindrical coordinates ρ, ϕ, z

$$\begin{aligned}
\nabla &= \hat{\mathbf{e}}_\rho \frac{\partial}{\partial \rho} + \hat{\mathbf{e}}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\
\nabla \cdot \mathbf{A} &= \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_\rho \left[\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] + \hat{\mathbf{e}}_\phi \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] + \hat{\mathbf{e}}_z \frac{1}{\rho} \left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right] \\
\nabla^2 &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}$$

spherical polar coordinates r, θ, ϕ

$$\begin{aligned}
\nabla &= \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \\
\nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_r \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right] + \hat{\mathbf{e}}_\theta \left[\frac{1}{r \sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right] + \hat{\mathbf{e}}_\phi \frac{1}{r} \left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \\
\nabla^2 &= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2} \\
& \left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \equiv \frac{1}{r} \frac{\partial^2}{\partial r^2} r \right]
\end{aligned}$$

Legendre polynomials

The Legendre polynomials of order of ℓ, $P_\ell(x)$, are given by the Rodrigues formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell,$$

where ℓ is a nonnegative integer. The first few Legendre polynomials are

$$P_0(x) = 1,\; P_1(x) = x,\; P_2(x) = \frac{1}{2}(3x^2-1),\; P_3(x) = \frac{1}{2}(5x^3-3x),\; \ldots$$

Legendre polynomials give an expansion of what is essentially the potential at $\mathbf{x}$ induced by a point charge at $\mathbf{x}'$:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta),$$

where $r_<$ ($r_>$) is the smaller (larger) of $|\mathbf{x}|$ and $|\mathbf{x}'|$, and θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{x}'$.