

**Preliminary Exam: Electromagnetism, Thursday January 12, 2017. 9:00-12:00**

Answer a total of any **THREE** out of the four questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

1. A sphere of radius  $R$  carries a charge  $Q$  distributed uniformly over the surface, with density  $\sigma = Q/4\pi R^2$ . This shell of charge is rotating at an angular speed  $\omega$  about a  $z$ -axis through the center of the sphere.
  - (a) Calculate the magnetic moment associated with this system. (Hint: The magnetic moment of a planar loop carrying a current  $I$  is  $IA$  where  $A$  is the area enclosed by the loop.)
  - (b) Evaluate the magnetic field at a point on the  $z$ -axis far from the sphere.
  - (c) Generalize the result in (b) to obtain the magnetic  $\mathbf{B}$  field at a point  $\mathbf{r}$  far outside the sphere.
2. A perfect electric dipole  $\mathbf{p}$  is located near a grounded conducting sphere of radius  $R$ . The vector connecting the center of the sphere and the dipole is  $\mathbf{r}_0$ , where  $r_0 = |\mathbf{r}_0| > R$ . The Green's function  $G$  associated with a unit point charge in the presence of such a conducting sphere is

$$4\pi\epsilon_0 G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{|\mathbf{r} - \mathbf{r}_0|} - \frac{Rr_0}{|r_0^2\mathbf{r} - R^2\mathbf{r}_0|}$$

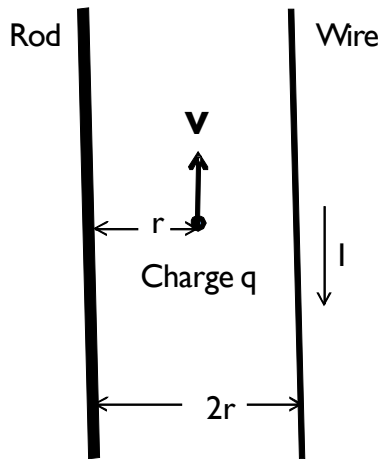
where  $\mathbf{r}_0$  is the location of the unit charge.

- (a) Based on the above Green's function, identify the location  $\mathbf{r}'$  and charge  $q'$  of the image of a point charge  $q$  at  $\mathbf{r}$ .

Expand this formula for  $d\mathbf{r}'$  to leading order in  $d\mathbf{r}$  about  $\mathbf{r}$ .

- (b) Using these results, give an expression for the image  $\mathbf{p}'$  of a dipole  $\mathbf{p}$  located at position  $\mathbf{r}_0$ . At what point  $\mathbf{r}'_0$  is this image dipole located?
- (c) Write down an expression for the electric potential from a single dipole in free space and hence obtain the electric potential everywhere outside the sphere by combining the above results.

3. A thin conducting sphere has a charge  $Q$  and radius  $a$ .
- (a i) What is the capacitance of the sphere (relative to a reference plane at infinite distance)?
- (a ii) What is the energy density of the electric field at a distance  $r$  from the center of the sphere, for both  $r < a$  and  $r > a$ .
- (a iii) What is the total energy of the electric field? Express the result in terms of the capacitance obtained in part (a i) and the potential  $V_0$  on the surface of the sphere.
- (b) A potential  $V$  is established between inner (radius  $a$ ) and outer (radius  $b$ ) shells of thin concentric conducting spheres. What is the radius of the inner sphere (i.e. the ratio  $a/b$ ) such that the electric field on its surface is minimum. Express this minimum value in terms of  $V$  and  $b$ .
4. Consider a thin linear rod with a uniform charge distribution of  $\lambda$  per unit length, and a wire of radius  $a$  that carries a current  $I$  parallel to the rod as shown in the figure. The separation between the rod and the wire is  $2r$ . A particle with charge  $q$  is traveling in the plane made by the rod and the wire with a velocity  $\mathbf{v}$  parallel to both rod and wire at a point equidistant from the rod and the wire.



- (a) Find the forces acting on the above charge when it is placed at a distance  $r$  from the rod.
- (b) Find the velocity  $\mathbf{v}$  (in terms of  $\lambda$  and  $I$ ) for which the particle will travel in a straight line parallel to the wire.
- (c) Now suppose the current  $I$  in the wire is increasing at a constant rate  $dI/dt$ . What is the electromagnetic force (magnitude and direction) on the particle under the conditions solved for in part (b)?

# Vector Formulas

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla \psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\nabla \cdot (\psi \mathbf{a}) = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

If  $\mathbf{x}$  is the coordinate of a point with respect to some origin, with magnitude  $r = |\mathbf{x}|$ ,  $\mathbf{n} = \mathbf{x}/r$  is a unit radial vector, and  $f(r)$  is a well-behaved function of  $r$ , then

$$\nabla \cdot \mathbf{x} = 3$$

$$\nabla \times \mathbf{x} = 0$$

$$\nabla \cdot [\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \quad \nabla \times [\mathbf{n}f(r)] = 0$$

$$(\mathbf{a} \cdot \nabla)\mathbf{n}f(r) = \frac{f(r)}{r} [\mathbf{a} - \mathbf{n}(\mathbf{a} \cdot \mathbf{n})] + \mathbf{n}(\mathbf{a} \cdot \mathbf{n}) \frac{\partial f}{\partial r}$$

$$\nabla(\mathbf{x} \cdot \mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla \cdot \mathbf{a}) + i(\mathbf{L} \times \mathbf{a})$$

where  $\mathbf{L} = \frac{1}{i}(\mathbf{x} \times \nabla)$  is the angular-momentum operator.

# Theorems from Vector Calculus

In the following  $\phi$ ,  $\psi$ , and  $\mathbf{A}$  are well-behaved scalar or vector functions,  $V$  is a three-dimensional volume with volume element  $d^3x$ ,  $S$  is a closed two-dimensional surface bounding  $V$ , with area element  $da$  and unit outward normal  $\mathbf{n}$  at  $da$ .

$$\int_V \nabla \cdot \mathbf{A} d^3x = \int_S \mathbf{A} \cdot \mathbf{n} da \quad (\text{Divergence theorem})$$

$$\int_V \nabla \psi d^3x = \int_S \psi \mathbf{n} da$$

$$\int_V \nabla \times \mathbf{A} d^3x = \int_S \mathbf{n} \times \mathbf{A} da$$

$$\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d^3x = \int_S \phi \mathbf{n} \cdot \nabla \psi da \quad (\text{Green's first identity})$$

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) d^3x = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} da \quad (\text{Green's theorem})$$

In the following  $S$  is an open surface and  $C$  is the contour bounding it, with line element  $d\mathbf{l}$ . The normal  $\mathbf{n}$  to  $S$  is defined by the right-hand-screw rule in relation to the sense of the line integral around  $C$ .

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} da = \oint_C \mathbf{A} \cdot d\mathbf{l} \quad (\text{Stokes's theorem})$$

$$\int_S \mathbf{n} \times \nabla \psi da = \oint_C \psi d\mathbf{l}$$

# Explicit Forms of Vector Operations

Let  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and  $A_1, A_2, A_3$  be the corresponding components of  $\mathbf{A}$ . Then

*Cartesian*  
( $x_1, x_2, x_3 = x, y, z$ )

$$\begin{aligned}\nabla\psi &= \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3} \\ \nabla \cdot \mathbf{A} &= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \\ \nabla \times \mathbf{A} &= \mathbf{e}_1 \left( \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) + \mathbf{e}_2 \left( \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) + \mathbf{e}_3 \left( \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right) \\ \nabla^2\psi &= \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}\end{aligned}$$


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*Cylindrical*  
( $\rho, \phi, z$ )

$$\begin{aligned}\nabla\psi &= \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho} \frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z} \\ \nabla \cdot \mathbf{A} &= \frac{1}{\rho} \frac{\partial}{\partial\rho} (\rho A_1) + \frac{1}{\rho} \frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z} \\ \nabla \times \mathbf{A} &= \mathbf{e}_1 \left( \frac{1}{\rho} \frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z} \right) + \mathbf{e}_2 \left( \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho} \right) + \mathbf{e}_3 \frac{1}{\rho} \left( \frac{\partial}{\partial\rho} (\rho A_2) - \frac{\partial A_1}{\partial\phi} \right) \\ \nabla^2\psi &= \frac{1}{\rho} \frac{\partial}{\partial\rho} \left( \rho \frac{\partial\psi}{\partial\rho} \right) + \frac{1}{\rho^2} \frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\end{aligned}$$


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*Spherical*  
( $r, \theta, \phi$ )

$$\begin{aligned}\nabla\psi &= \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r} \frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r \sin\theta} \frac{\partial\psi}{\partial\phi} \\ \nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_1) + \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta A_2) + \frac{1}{r \sin\theta} \frac{\partial A_3}{\partial\phi} \\ \nabla \times \mathbf{A} &= \mathbf{e}_1 \frac{1}{r \sin\theta} \left[ \frac{\partial}{\partial\theta} (\sin\theta A_3) - \frac{\partial A_2}{\partial\phi} \right] \\ &\quad + \mathbf{e}_2 \left[ \frac{1}{r \sin\theta} \frac{\partial A_1}{\partial\phi} - \frac{1}{r} \frac{\partial}{\partial r} (r A_3) \right] + \mathbf{e}_3 \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_2) - \frac{\partial A_1}{\partial\theta} \right] \\ \nabla^2\psi &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial\psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} \\ &\quad \left[ \text{Note that } \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial\psi}{\partial r} \right) \equiv \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi). \right]\end{aligned}$$