Preliminary Exam: Electromagnetism, Thursday January 15, 2015, 9:00-12:00

Answer a total of any **THREE** out of the four questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

Remark: SI units are used in the formulas quoted below. If you prefer to work with Gaussian units, replace ϵ_0 by $(4\pi)^{-1}$.

Problem 1:

Consider a point charge q inside a conducting grounded sphere of radius R at the distance $a < R$ from the center of the sphere.

(a) Determine the Greens function $G_D(\vec{x}, \vec{x}\,')$ for this Dirichlet-type problem.

(b) Derive the potential $\phi(\vec{x})$ and the electric field $\vec{E}(\vec{x})$ inside the sphere.

(c) Determine the induced charge density per unit area, $\sigma(\vec{x})$, on the sphere, and compute the total induced charge $Q_{\text{ind}} = \int_S \sigma(\vec{x})\, \mathrm{d}a$ where S denotes the area of the sphere.

Hint: Use the convention $\triangle_x G_D(\vec{x}, \vec{x}\,') = -4\pi\, \delta^{(3)}(\vec{x} - \vec{x}\,')$, where the solution of a Dirichlet-type problem is given by $\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int\limits_V \rho(\vec{x}\,')\, G_D(\vec{x}, \vec{x}\,') \mathrm{d}^3x' - \frac{1}{4\pi} \int\limits_S \phi(\vec{x}\,')\, \frac{\partial G_D(\vec{x}, \vec{x}\,')}{\partial n'}\, \mathrm{d}a'$.

Problem 2:

Consider a homogeneous and isotropic conductor with conductivity σ, permittivity ϵ, permeability μ, and volume V_{cond}. Let a small volume $V_0 \ll V_{\text{cond}}$ inside the conductor contain a charge distribution $\rho_0(\vec{x}) \neq 0$ at the time $t = 0$.

(a) Derive the expression $\rho(\vec{x}, t)$ describing the charge distribution inside the volume V_0 for times $t > 0$, assuming that $\rho(\vec{x}, t)$ varies with time sufficiently slowly to neglect retardation effects.

(b) Determine the current $I(t)$ flowing out of the volume V_0. Calculate the total charge $Q = \int\limits_0^{\infty} \mathrm{d}t\, I(t)$ that flows out of the volume V_0. Where does Q end up "after" $t \to \infty$?

(c) Using the quantities specified in the problem, find the dimensionless ratio $x \ll 1$ which justifies that "things vary with time slowly" in part (a).

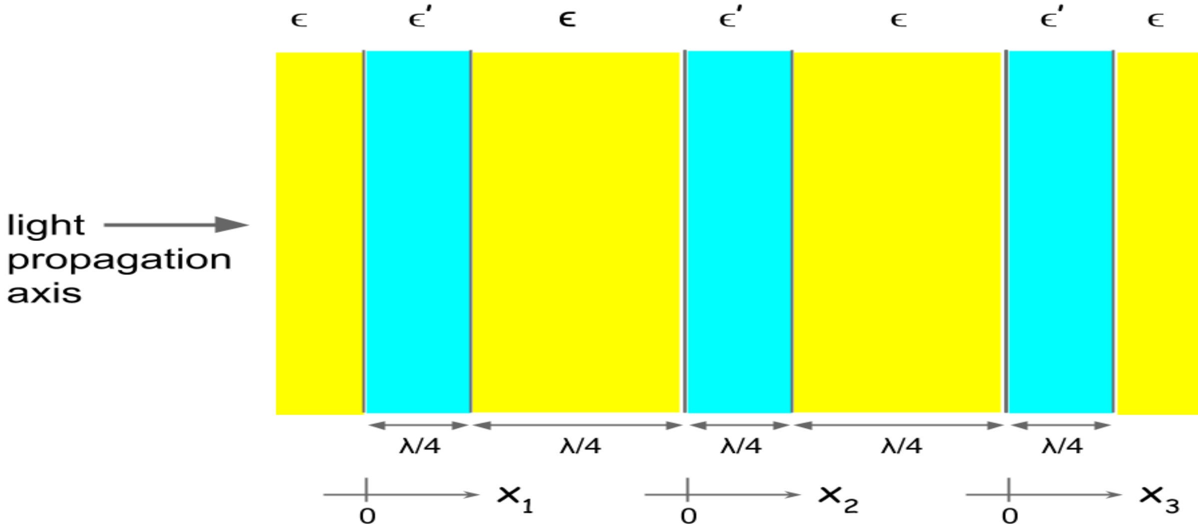
Problem 3:

Two long cylindrical conductors of radius a_1 and a_2 are parallel and separated by a distance d where d is large compared to a_1, a_2. Show that the capacitance per unit length is approximately given by

$$C \simeq \frac{\pi\epsilon_0}{\ln(d/a)} \quad \text{where} \quad a = (a_1 a_2)^{1/2}.$$

Problem 4:

Consider a multi-layer dielectric consisting of alternating layers of low (ϵ) and high (ϵ') refractive index, as shown in the figure. The layers are made of materials with the same $\mu = \mu_0$ but with different dielectric constants ϵ. The layer thickness of all layers is exactly one quarter of one wavelength at frequency f_0.



Let the components of the electric fields in the two layers that share the interface at $x = 0$ be written as:

$$E_1 = a_1 e^{ikx} + b_1 e^{-ikx} \quad E_2 = a_2 e^{ikx} + b_2 e^{-ikx} \quad E_3 = a_3 e^{ikx} + b_3 e^{-ikx}$$
$$E_1' = c_1 e^{ik'x} + d_1 e^{-ik'x} \quad E_2' = c_2 e^{ik'x} + d_2 e^{-ik'x} \quad E_3' = c_3 e^{ik'x} + d_3 e^{-ik'x}$$

where k is the wavenumber of frequency f_0 in the fast dielectric and k' is the wavenumber of frequency f_0 in the slow dielectric.

(a) Write down the appropriate matching conditions for the fields $\vec{E}$, $\vec{E}'$ and $\vec{B}$, $\vec{B}'$ for normal incidence waves at the $x = 0$ interface.

(b) Apply the rules from part (a) to obtain amplitudes b and c in terms of a and d. Rearrange your result into a matrix equation giving column vector $\binom{a_1}{b_1}$ in terms of vector $\binom{c_1}{d_1}$.

(c) How does the result of part (b) change if the order of the two dielectrics is reversed? Use this fact to obtain relations between $\binom{c_1}{d_1}$ and $\binom{a_2}{b_2}$ at the second interface at $x_1 > 0$.

(d) Combine parts (b) and (c) to get a matrix relation between $\binom{a_1}{b_1}$ and $\binom{a_2}{b_2}$. Successive powers of this matrix gives the amplitude of the light as it propagates into layer 3, 4 ... Diagonalize this matrix and find the eigenvalues and eigenvectors of light transport in this structure. Which mode corresponds to light entering from the left as shown? Is this a high reflectivity coating or an antireflection coating? Why or why not?