Classical Mechanics/Electricity and Magnetism
General Exam
August 23, 2006
09:00 - 15:00 in P-121

Answer three (3) questions from each of the two (2) sections for a total of six (6) solutions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented separately in an answer book, or on consecutively numbered sheets of paper stapled together. Make sure you clearly indicate who you are, and what is the problem you are answering. Double-check that you include everything you want graded, and nothing else.

The backside of this page has a copy of the vector calculus formulas from the back of Jackson.

Section 1 — mostly mechanics

1. A disc rolls without slipping down an inclined plane A with the angle ϕ and a hoop rolls without slipping down another inclined plane B with the angle ψ. The lengths of the two planes are the same. The disc and the hoop start at the top of the respective planes simultaneously, and get to the bottom also at the same time. Find a relation between the angles ϕ and ψ.

2. Consider two particles of mass m each, connected by springs of force constant k, between two rigid walls as shown in Figure 1. Assuming the particles are constrained to move only in the x direction, calculate the oscillation frequencies of the system of two particles for small displacements in the x direction.

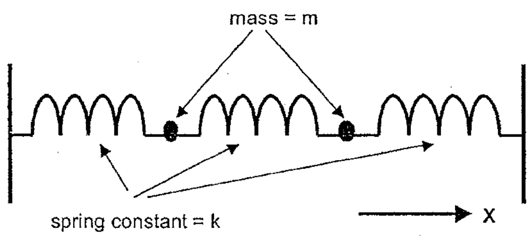

Figure 1: Drawing for problem 2.

3.

(a) A particle of mass m and electric charge e moves in a plane under the influence of a central force potential $V(r)$ and a constant uniform magnetic field $\mathbf{B}$ perpendicular to the plane. This magnetic field may be derived from the static vector potential $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$. Find the Hamiltonian using the coordinates in the observer's inertial frame.

(b) Find the Hamiltonian using coordinates rotating relative to the previous coordinate system about an axis perpendicular to the plane with an angular rate $\omega = -\frac{eB}{2m}$. Is the Hamiltonian conserved? Explain.

4. A rocket ship leaves earth at a speed of $\frac{3}{5}c$. When a clock on the rocket says 1 hour has elapsed, the rocket ship sends a light signal back to earth.

(a) According to *earth* clocks, when was the signal sent?

(b) According to *earth* clocks, how long after the rocket left did the signal arrive back on earth?

(c) According to the *rocket* observer, how long after the rocket left did the signal arrive back to earth?

(d) Calculate the invariant interval $(c^2\Delta t^2 - \Delta x^2)$ between the following two events: event A - the rocket ship sends the signal; event B - the signal arrives on earth.

Section 2 — mostly E&M

5. A superconducting sphere of radius a is inserted slowly into a constant (in time) magnetic field that may also be regarded as constant (in space) over the length scale a. This idealized type-I superconductor does not allow any magnetic field inside. Find the steady-state magnetic field outside.

6. Consider an electron at rest (velocity $= 0$) at a distance d (at time $t = 0$) from an infinite conducting plane (Figure 2). Determine the equation of motion of this electron. What is the maximum kinetic energy of the electron? A moving electron produces a magnetic field. How does the magnetic field produced by this electron vary with time? Ignore short-range interactions and radiative damping.

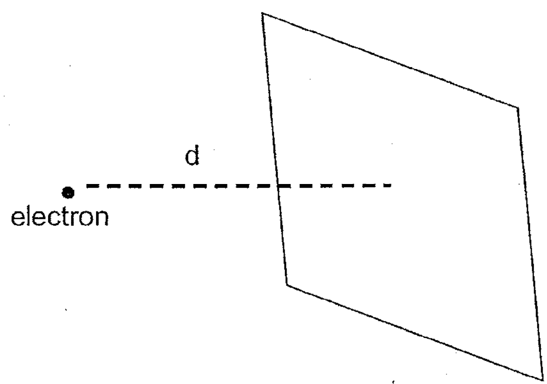

Figure 2: Drawing for problem 6.

7. A plane polarized electromagnetic wave of frequency ω in free space is incident normally on the flat surface of a medium of conductivity σ and dielectric constant ϵ.

(a) Calculate the amplitude and phase of the reflected wave relative to the incident wave.

(b) Discuss the limiting cases of a very poor and a very good conductor, and show that for a good conductor the reflectivity (ratio of reflected to incident intensity) is given by

$$R \simeq 1 - 2(\omega/c)\delta\,,$$

where δ is a quantity known as skin depth (and $\delta \to 0$ in the limit of infinite conductivity).

8. Take two very large conducting plates in the planes $z = 0$ and $z = d$, with empty space in between. Show that plane-wave like modes of the form $E(\mathrm{x}, t) = E(z)e^{i(kx-\omega t)}\hat{\mathrm{e}}_y$ with the polarization parallel to the plates may propagate between the plates, but only if the frequency exceeds a certain cutoff. HINT: You are most likely dead in the water if you try to memorize the general formalism for waveguides.

Explicit Forms of Vector Operations

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directior specified in the headings on the left, and A_1, A_2, A_3 be the corresponding corr ponents of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$