Classical Mechanics / Electricity and Magnetism

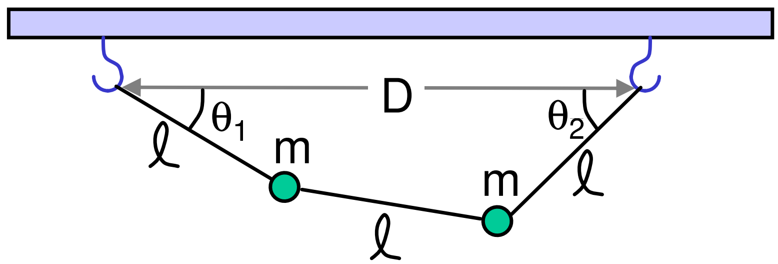
General Exam Questions for August 24, 2005

Instructions

Answer three questions from each of the two sections, for a total of six problems. Put each of your solutions in a separate answer book. Make sure that you label and sign your name on the cover of each book.

I. Classical Mechanics

1. Consider the 2-mass pendulum shown below, where the masses can only move in the plane of the figure. The links connecting the masses are massless strings of equal and fixed length.



   a) What are the values of θ_1 and θ_2 in the ground state?

   b) What is the frequency of small oscillations for the lowest-frequency mode of the system?

2. The Lagrangian for a system of one degree of freedom is written as

$$L = \frac{m}{2}\left(\dot{q}^2 \sin^2 \omega t + \dot{q} q \omega \sin 2\omega t + q^2 \omega^2\right)$$

   a) What is the corresponding Hamiltonian?

   b) Change to a new coordinate $Q = q \sin \omega t$, and find the new Lagrangian and Hamiltonian.

   c) Is the Hamiltonian conserved in your result for part (a)? part (b)? Explain any differences.

3. Consider a point particle of mass $m = 1$ moving in the xz plane under gravitational acceleration $\vec{g} = -\hat{z}$. The particle is confined to move along a curve $x(z)$ without friction. Find a curve $x(z)$ such that the downward component of the velocity $\dot{z}$ is a constant of the motion.

4. A star is orbiting in a circular orbit around an invisible companion at a constant angular velocity ω_0. A distant inertial observer located in the plane of the orbit observes the motion through the periodic modulation of the central frequency of a spectral line emitted by the star, of frequency $\omega_1 \gg \omega_0$. What is the time dependence $\omega_1(t)$ of the spectral line frequency, as a function of the parameters of the orbit (period T, radius R, etc.)? Define all quantities, without making any non-relativistic assumptions.

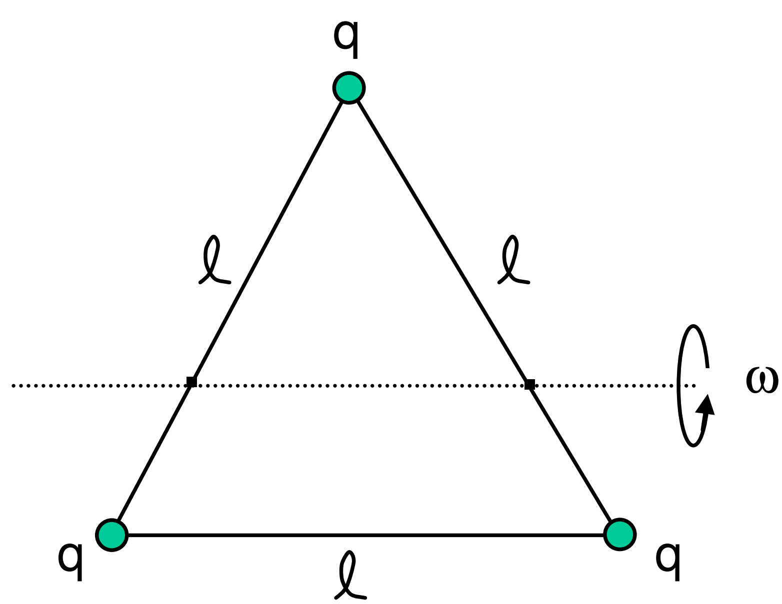
II. Electricity and Magnetism

1. A sphere of radius a has been fabricated from a material with a constant magnetization $\vec{M}$. What is the magnetic induction $\vec{B}$ inside the sphere?

2. Three equal charges q are fixed at the corners of an equilateral triangle. The triangle rotates at a frequency ω about an axis through its center parallel to one of its sides. The Cartesian moment Q_{hkm} of the instantaneous charge distribution is defined as

$$Q_{hkm}(t) = \frac{1}{Q} \sum_i q_i x_i^h y_i^k z_i^m$$

where the index i sums over the discrete charges q_i located at position (x_i, y_i, z_i), and Q is the total charge of the system.



a) Compute all moments Q_{hkm} with $h + k + m \leq 2$ which are non-zero, as a function of time t.

b) If this system acts as a source of an electromagnetic field, what frequencies are present in the spectrum of the radiation? Include only the contributions from the moments computed in part (a).

3. Consider an atom with the complex polarizability $\gamma(\omega) = \gamma'(\omega) + i\gamma''(\omega)$ with real functions $\gamma'(\omega)$ and $\gamma''(\omega)$. The atom exists in a external electromagnetic field given by $\vec{E}(\vec{x})e^{-i\omega t}$ and $\vec{B}(\vec{x})e^{-i\omega t}$ which satisfy the equations of motion for the free electromagnetic field. Assuming that the spatial variation of the external field is gradual on the scale of the atomic dimensions, it is easy to show that the average force that the field exerts on the atom is given by

$$F_i = -\frac{1}{2}\Re e[\gamma E_j^* \frac{\partial}{\partial x_i} E_j]$$

with an implied sum over j from 1 to 3. Evaluate $\vec{F}$ in the following two cases.

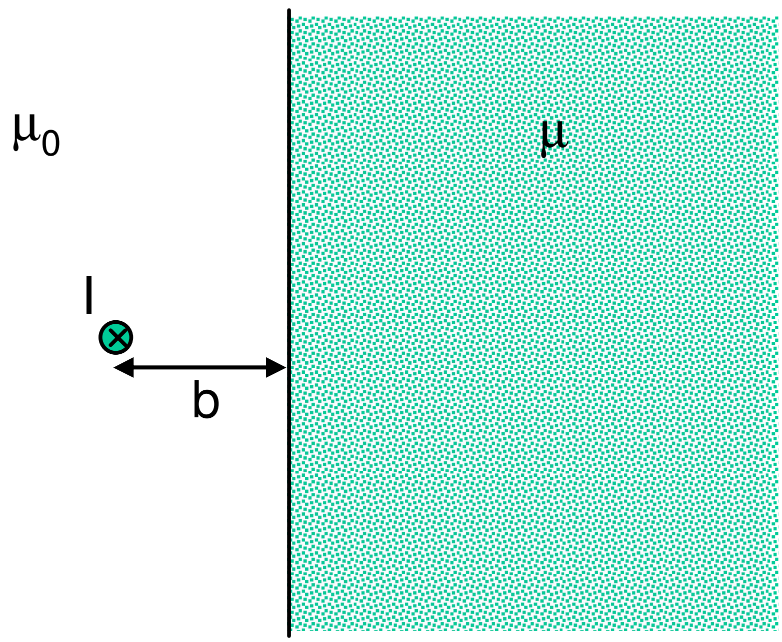
a) an atom in pure plane wave

b) an atom near the focus of a far off-resonance laser beam. The field in this case is of the form $\vec{E} = S(\vec{x})e^{i\vec{k}\cdot\vec{x}}\hat{\epsilon}$, where $S(\vec{x})$ changes little over a wavelength. Therefore, S can be taken to be real, and S^2 is proportional to the energy density of the laser beam. Moreover, far off resonance, $|\gamma'| \gg \gamma''$.

What is the magnitude of the force on the atom in each case? Which way does the force on the atom point for each?

Hint: Laser cooling of atoms corresponds to case (a), and laser trapping of atoms corresponds to case (b).

4. A long straight wire carrying a steady current I is placed parallel to a large thick slab of permability μ, as shown in the figure. The wire diameter is negligible.



a) Show that the field inside the slab is the same as what would be produced by the same wire carrying current $I' = I\frac{2\mu_o}{\mu_o+\mu}$ if the slab material filled all of space.

b) Show that the field outside the slab is equivalent to that produced by the physical current I and an image current $I'' = I\frac{\mu-\mu_o}{\mu+\mu_o}$ in the absence of the slab.

c) What is the force on the wire per unit length?

Hint: Solve this problem using the method of images, with an the image current located at the reflection of the physical wire through the interface plane. When solving for the field on the left side, introduce the image current and adjust its value to satisfy the matching conditions at the boundary. When solving for the field on the right side, the image current switches sides so that it sits on top of the real current, where its only effect is to change the apparent magnitude of I.