Classical Mechanics / Electricity and Magnetism

General Exam Questions for January, 2007

Instructions

Answer three questions from each of the two sections, for a total of six problems. Put each of your solutions in a separate answer book. Make sure that you label and sign your name on the cover of each book.

I. Classical Mechanics

1. A classical particle of mass m is moving in a velocity-dependent force field given by

$$\vec{F} = \vec{v} \times \vec{A} + B\vec{r}$$

in an inertial frame, where $\vec{A} = A\hat{z}$ and $B < 0$ are constants that do not depend on the coordinates or time.

a) What are the periods of all possible circular orbits in the xy plane observed in an inertial reference frame with the origin at $\vec{r} = 0$?

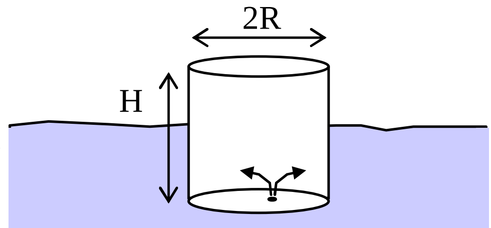
b) Show that this system can be transformed into a 3D harmonic oscillator by transforming to the coordinates of a rotating frame. What is the rotation angular velocity ω of the rotating frame? What is the rotation axis? *Hint:* The time derivative of a vector $\vec{Q}$ in the inertial frame is related to the same vector in the rotating frame $\vec{Q}\,'$ through the following formula

$$\mathbf{R}\frac{d\vec{Q}}{dt} = \frac{d\vec{Q}\,'}{dt} + \vec{\omega} \times \vec{Q}\,' ,$$

where $\mathbf{R}$ is the rotation operator connecting the two frames, as in $\vec{Q}\,' = \mathbf{R}\vec{Q}$.

c) What are the characteristic frequencies of the system in the rotating frame? What conditions on A and B must hold in order for the system to possess just one oscillation frequency?

2. An empty barrel of mass m and volume $V = \pi R^2 H$ is floating in an ideal fluid of constant density ρ. The sides and bottom of the barrel are of negligible thickness



and the top is open. At $t = 0$ a very small hole of area a is created in the bottom of the barrel, so that it begins to fill up and eventually sinks. The leak rate is slow enough that the barrel remains in static equilibrium (its velocity and acceleration are negligible compared with all other scales in the problem).

a) What is the difference in the fluid levels Δh inside and outside the barrel? Show that this difference is constant until the barrel sinks.

b) What is the difference ΔP in the fluid pressure across the bottom of the barrel, ie. between the inside and outside surfaces of the barrel floor?

c) What is the velocity of the fluid in the hole? *Hint:* Consider the net momentum per unit time coming through the hole and set that equal to the net force. The flow is stead-state.

d) How much time elapses between the instant the leak appears and the time the barrel is submerged?

3. A classical particle moves in a plane subject to the Lagrangian

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + qBy\dot{x} + qEy$$

where E and B are constants.

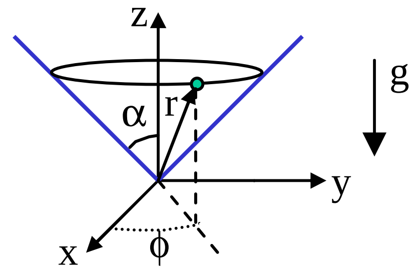
a) Find two constants of the motion.

b) Find the complete solution $x(t)$ and $y(t)$, starting with $x(0) = 0$, $y(0) = 0$, $\dot{x}(0) = -E/B$ and $\dot{y}(0) = v$. (*Hint:* You should be able to do this by making use of the constants of the motion, without solving any second-order differential equations.)

Useful integral:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

4. Consider a particle of mass m moving on th face of a cone with half-angle α oriented wi vertex pointing downward in a uniform gr. tional field with acceleration g, as shown.



a) Set up a Lagrangian using spherical coordinates r and ϕ.

b) Identify a conserved angular momentum and take its value to be ℓ.

c) Derive a relation of the following form, and find $U_{eff}(r')$.

$$t(r) = \int^{r} \sqrt{\frac{m}{2[E - U_{eff}(r')]}}\, dr'$$

d) Suppose the particle has a very large energy E and a fixed ℓ. What are (approximately) the maximum and minimum allowed values of r?

e) Find $\dot{\phi}(t)$ for a motion having constant $r = r_0$. Express your answer in terms of g, r_0, and α.

II. Electricity and Magnetism

1. A sphere of radius R contains a charge Q uniformly distributed within its interior.
   a) What is the electric potential $\Phi(r)$ for all r?
   b) What is the electrostatic self-energy of the sphere?
   c) What happens to the self-energy in the limit where R goes to zero?

2. A steady stream of ions of mass m and charge q emerges from a source in vacuum, and is accelerated such that each ion receives the same energy increment $\Delta E = E_a \gg mc^2$. The accelerated ions then enter a storage trap containing a uniform magnetic field $\vec{B}$ between two parallel planar magnetic poles, where they circulate indefinitely.
   a) What is the radius r of the ion orbit in the trap?
   b) If the circulating ions carry an electrical current I, what is the density of ions per unit length of the beam in a frame co-moving with the ions?
   c) If an ion enters the trap on the symmetry plane between the two magnetic poles, but has a small momentum component $p_z \ll E_a/c$ in the direction of $\vec{B}$, how many turns around the trap will it make before it collides with the pole a height h above the mid-plane?
   d) Even if they do not collide with anything, the ions in the trap gradually lose energy and slow down. Where does the energy lost by the ions go? What is the process by which this occurs?

3. Two parallel infinite sheets at $x = \pm a$ are configured with the electrostatic potentials

$$\begin{aligned}\Phi(-a, y, z) &= \Phi_0 + \Phi_1 \sin k_1 z \ , \\ \Phi(+a, y, z) &= \Phi_0 + \Phi_1 \sin k_2 z \ .\end{aligned}$$

with vacuum between them. What is the potential $\Phi(x, y, z)$ in the region between the plates?

4. An infinitely long insulating filament with linear charge density λ lies at rest along the z-axis.

   a) Find the electrostatic field $\vec{E}$ at a point P a distance x_0 away from the origin along the x-axis.

   b) At $t = 0$, the wire suddenly starts to move with constant velocity v in the positive z-direction. Assuming the wire is infinitely thin, write down an expression for the current density $\vec{j}$ arising from the motion.

   c) Using the following formula for the retarded potential calculate $A_z(x_0, t)$. Give its value for $t > x_0/c$ and for $t < x_0/c$.

$$\vec{A}\,(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\vec{j}\left[\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right]}{|\vec{x} - \vec{x}'|}$$

   d) Because of cylindrical symmetry you know $A_z(\rho, t)$ with ρ the radial coordinate in cylindrical coordinates. Find $\vec{B}(\rho, t)$ as $t \to \infty$. Does your value agree with your intuitive expectation from Ampere's law?

Useful integral:

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = ln \left| \frac{x + \sqrt{x^2 + a^2}}{a} \right| \tag{1}$$