Preliminary Exam: Classical Mechanics, Monday August 20, 2018. 9:00-12:00

Answer a total of any **THREE** out of the four questions. Put the solution to each problem in a separate blue book and put the number of the problem and your name on the front of each book. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

1. **Normal modes**

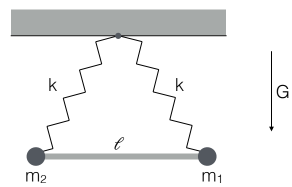

Figure 1: Two springs and a bar

Consider the system depicted in the Figure, consisting of two different masses m_1 and m_2, hanging from one point off the ceiling by equal massless springs with spring constant k and zero unexpanded length (i.e., the length that the spring has if lying unattached on a table). Assume that the system is two-dimensional, i.e., it lives in the vertical plane (the plane in the Figure). In addition, the two masses are attached at the ends of a massless rod with length ℓ. For this system,

(**a**) Show why the system has three degrees of freedom and give the Lagrangian and the Lagrange equations of motion. *(Hint: A good choice of variables: the coordinates of the middle of the bar and the angle around this middle.)*

(**b**) Find the equilibrium point of this system if gravity goes downward. (There are 2 positions - which is stable?) Then find frequencies and eigenmodes for all small oscillations. (*Note: Some of the eigenmodes look complicated. Just give a recipe of how to calculate them and describe or draw them qualitatively.)*

2. **Falling rod**

A uniform thin rod of length ℓ and mass m stands straight up against gravity (acceleration g) on a slippery (frictionless) floor. This situation is unstable, and a slightest perturbation causes the rod to fall.

(**a**) The center of mass of the rod falls straight down. Give a sound reason for this.

The position and orientation of the rod may therefore be completely specified using the spherical polar coordinates θ and ϕ, taking the z direction up against gravity. When the angles θ and ϕ change, the rod rotates. It may be shown that the square of the component of the angular velocity vector perpendicular to the rod equals

$$\omega_\perp^2 = \dot\theta^2 + \sin^2\theta\,\dot\phi^2.$$

(**b**) Show that the kinetic and potential energies are (or may be chosen to be)

$$T = \tfrac{1}{8}m\ell^2\sin^2\theta\,\dot\theta^2 + \tfrac{1}{24}m\ell^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2), \quad U = \tfrac{1}{2}mg\ell\cos\theta\,.$$

(**c**) Show that the rod stays in a plane when it falls.

(**d**) What is the downward speed of the center of mass when the rod slams to the floor? Compare this with the velocity it would have had if the rod had dropped the same distance with nothing in the way. If these two velocities are different, where did the extra energy go in the first case?

3. **Rotating dumbbell**

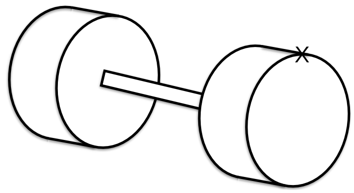

Figure 2: A dumbbell: two massive cylinders connected by a massless bar.

Two cylinders of mass m each, with radius r and height h are connected by a massless bar as in the figure. The centers of the cylinders are separated by L.

a) What are the principal axes of this setup through the center of mass of the dumbbell? (You can describe them or make a simple drawing.) Calculate the moment of inertia tensor through the center of mass. (On the way, show that the moment of inertia through the center of a cylinder and around an axis perpendicular to the symmetry axis is $m(3r^2 + h^2)/12$.)

b) What is the moment of inertia through the point marked with an "x" in the figure (assume the mark is on the upper outer edge of one of the cylinders), around the horizontal axis that marks the tangent to the cylinder rim?

c) If one would keep the point marked with an "x" fixed (for example, by holding on to it at this point), but would let the rest of the dumbbell fall, what is the kinetic energy of the system when the face of the cylinder is horizontal? (That is, the dumbbell bar is horizontal at the start of the movement and vertical at the end.) What is the angular momentum around its movement axis at that time (this is the same axis as in part (b))? (Assume that $L+h > 2r$.)

d) If one replaces the round cylinders with elliptical ones (where the long axes of the ellipse of both cylinders are parallel to each other), can you draw or describe the principal axes through the center of mass in this case? Around which of the axes is the moment of inertia largest, around which is it smallest? (Assume that both ellipse radii are considerably smaller than the distance between the cylinders?) Without proof or derivation, rotation around which of those axes is stable and which unstable?

Reminder: The moment of inertia tensor for N particles can be described as

$$I_{\alpha\beta} = \sum_i^N m_i \left(|\vec{r}_i|^2 \delta_{\alpha\beta} - r_{i\alpha} r_{i\beta} \right) .$$

4. **Particle moving on cone surface**

Suppose a particle of mass m is somehow constrained to move on a cone, $z = \lambda\rho$ in cylindrical coordinates, with $z > 0$, and is also subject to a radially symmetric potential energy $U(\rho, z)$.

(**a**) Show that the z component of the angular momentum is a constant of the motion.

(**b**) In light of the result of part (a), there will be an effective 1D motion for the radial coordinate ρ. What is the corresponding effective force $F(\rho)$?