Preliminary Exam: Classical Mechanics, Monday August 22, 2016. 9:00-12:00

Answer a total of any **THREE** out of the four questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

1. In Figure (1) a cube of mass m with a side of length $2b$ sits on top of a fixed rubber horizontal cylinder of radius r, with the earth's gravity acting vertically downwards. The cube cannot slip on the cylinder, but it can rock from side to side. Assume that the cube is initially balanced on the cylinder with its center of mass C directly above the center of the cylinder O.

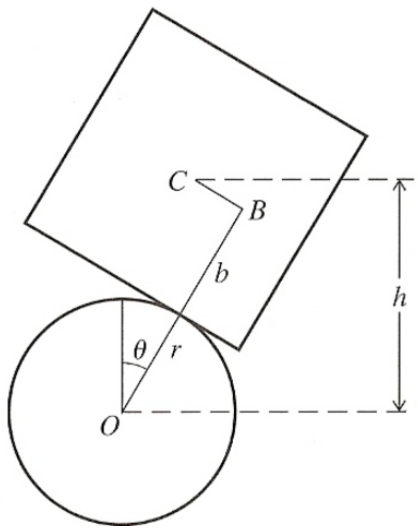

Figure 1: Cube on top of a cylinder

(a) Show that the the Lagrangian for this system is given by

$$\mathcal{L} = \frac{m}{2}\left(\frac{5b^2}{3} + r^2\theta^2\right)\dot{\theta}^2 - mg\left[(r+b)\cos\theta + r\theta\sin\theta\right].$$

(b) Find the Lagrange equations of motion.

(c) Find any possible positions of equilibrium.

(d) Choose any one of the possible positions of equilibrium from part (c), and determine whether it is stable. If it is, find the frequency of small oscillations about equilibrium. (Assume that the corner of the cube never touches the sphere.)

2. (a) Show that if a particle under the influence of a central force $f(r)$ has an orbit which is a circle passing through the point of attraction, then the force is a power law with $|f| \propto r^{-5}$.

(b) Assuming the potential is defined so that $U(\infty) = 0$, show that for this particular orbit $E = 0$.

(c) In terms of the diameter and the angular momentum, find the period of the orbit.

Hint: For part (a), try to write the orbit as a function of the angle. Remember, that in a central force $f(r)$, the angular momentum L, radius r, and angle θ are related by

$$\frac{L}{r^2}\frac{d}{d\theta}\left(\frac{L}{mr^2}\frac{dr}{d\theta}\right) - \frac{L^2}{mr^3} = f(r).$$

3. A long thin uniform bar of mass M and length L is hung from a fixed frictionless horizontal axis at A. The earth's gravity acts vertically downwards. The rod is hanging motionless in the equilibrium position when an instantaneous horizontal impulse $\mathbf{J}$ in the plane of Figure (2) is applied to the bar at B at a distance a from the axis located at A.

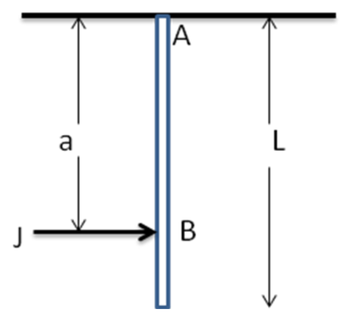

Figure 2: Impulse applied to a bar

(a) Determine the angular velocity of the bar immediately after the impulse is applied.

(b) Determine the impulse $\mathbf{J_1}$ at axis A as a result of the impulse $\mathbf{J}$.

4. In Figure (3) a uniform rod of mass m and length L is supported at its ends by two massless springs with spring constants k_1 and k_2 respectively, and is horizontal when at rest. The earth's gravity acts vertically downwards. Consider small amplitude motions y_1 and y_2 and assume that the motions of the springs are restricted to the vertical plane.

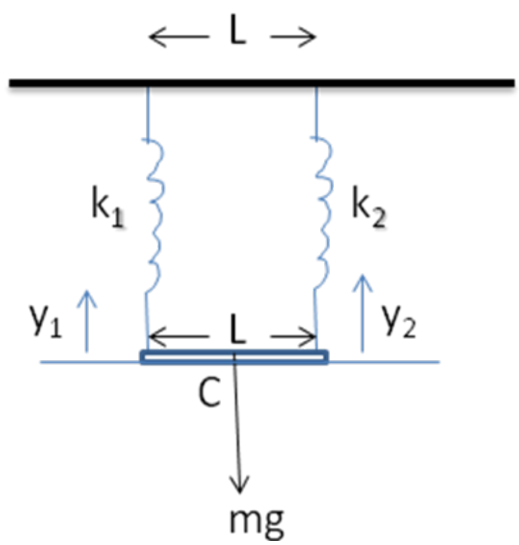

Figure 3: Rod with springs

(a) Consider the special case $k_1 = k_2$. Determine the frequencies of normal modes and describe the corresponding motions.

(b) Consider the general case when k_1 and k_2 are not equal and determine the normal mode frequencies. Hint: If y_1 and y_2 are different, the rod will experience a net torque, and hence can rotate about its center of mass at C with angle of rotation $\theta \sim (y_1 - y_2)/L$.