

Preliminary Exam: Classical Mechanics, Monday January 12, 2015, 9:00-12:00}

Answer a total of any THREE out of the four questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

Problem 1

Classical

A system in uniform gravity, consisting of two masses m , each carrying a charge q , is arranged as in the figure, that is, the arms of length ℓ with the masses can move in θ and ϕ direction independently. (The interaction between the charges is $V_{12} = \frac{q^2}{r_{12}}$, where r_{12} is the distance between the masses.) Find the Lagrangian and the equilibrium (the assumption that $\phi_{1,0} - \phi_{2,0} = \pi$ and $\theta_{1,0} = \theta_{2,0} \equiv \theta$ can be made from symmetry without proof). Show that the equilibrium condition is

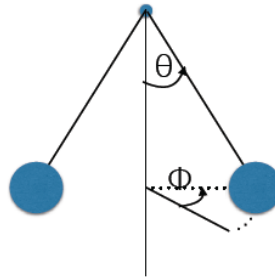
$$\frac{4mg\ell^2}{q^2} \tan^3 \theta = 1 + \tan^2 \theta.$$

(You do not need to solve this for θ explicitly.) Then show that for the potential we have for the deviations from the equilibrium position $\{\delta\theta_1, \delta\theta_2, \delta\phi_1, \delta\phi_2\} \equiv \{\eta_i\}$

$$\left. \frac{\partial^2 V}{\partial \eta_i \partial \eta_j} \right|_{\text{equilibrium}} = \begin{pmatrix} u+v & v & 0 & 0 \\ v & u+v & 0 & 0 \\ 0 & 0 & w & -w \\ 0 & 0 & -w & w \end{pmatrix}.$$

What are the constants u, v , and w ?

Then, show the eigenmodes (i.e., frequencies and eigenvectors) for small deviations from the equilibrium.



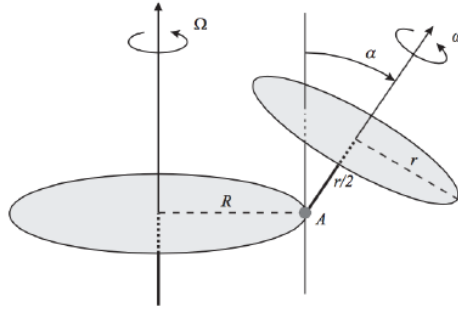
You can use (any or none or all of)

$$\begin{aligned} \sin(x + \delta x) &\approx \sin x + \delta x \cos x - \frac{1}{2}(\delta x)^2 \sin x, \\ \cos(x + \delta x) &\approx \cos x - \delta x \sin x - \frac{1}{2}(\delta x)^2 \cos x, \\ \tan(x + \delta x) &\approx \tan x + \delta x (\tan^2 x + 1) + (\delta x)^2 (\tan^3(x) + \tan(x)), \\ \frac{1}{r_{12}} &\approx \frac{1}{2\ell \sin \theta} \left(1 + \mathcal{O}(\delta\theta_i, \delta\phi_i) + \frac{1}{8} \left(\left(\frac{2}{\sin^2 \theta} - 1 \right) (\delta\theta_1 + \delta\theta_2)^2 + (\delta\phi_1 - \delta\phi_2)^2 \right) \right), \end{aligned}$$

Problem 2

Classical

A top in a uniform gravitational field consists of a homogeneous, circular disc of radius r , and a massless axle of length $r/2$ perpendicular to the disc through its center. The axle is connected by a joint to a point A on the periphery of a horizontal circular merry-go-round of radius R , so that the axle of the top can pivot with negligible friction in a vertical plane containing the axle of the merry-go-round. The merry-go-round has the constant angular velocity Ω about its axle, and the top has the constant angular velocity ω , relative to the merry-go-round, about its axle. Find the equilibrium angle α between the axle of the top and the vertical.



A few hints: This problem might be easiest if one uses (and compares) a top-fixed and a space-fixed coordinate system. How do they convert (especially since one is time dependent, the other is not)? Show during your calculation that the absolute value of the torque on the top is given by

$$|\mathbf{N}| = \frac{r}{2}mg \sin \alpha + \frac{r}{2}m\Omega^2\left(R + \frac{r}{2} \sin \alpha\right) \cos \alpha.$$

Since you are looking for the equilibrium angle α , it is fine to set $\dot{\alpha} = 0$ throughout the problem.

Problem 3:

A uniform rigid rod AB of mass M and length L is free to rotate smoothly (in a vertical plane χ) about a horizontal axis through A (fixed).

- When the rod is hanging vertically at rest with B below A, a bullet of mass m , moving horizontally in χ with speed v perpendicular to the rod, hits the rod and gets embedded in it at B. What is the *initial angular speed* of the rod immediately after this impact? Briefly justify your answer explaining why your approach is valid.
- If the impact is *barely sufficient* to make the rod horizontal, find the *initial speed* v of the bullet, before the impact (in terms of l and m). Again, justify your answer explaining why your approach is valid.
- If the hinge at C is not smooth, are your answers to (a) and (b) still valid? Explain briefly. (No calculations here.)

Problem 4:

Three identical objects, each of mass of m , are connected by springs of spring constant k as shown in figure. The motion is confined to one dimension. At $t = 0$, the masses are at rest at their equilibrium positions. Mass A is subjected to a force of $F = f \cos(\omega t)$, $t > 0$. Calculate the motion of mass C. All surfaces are frictionless.

