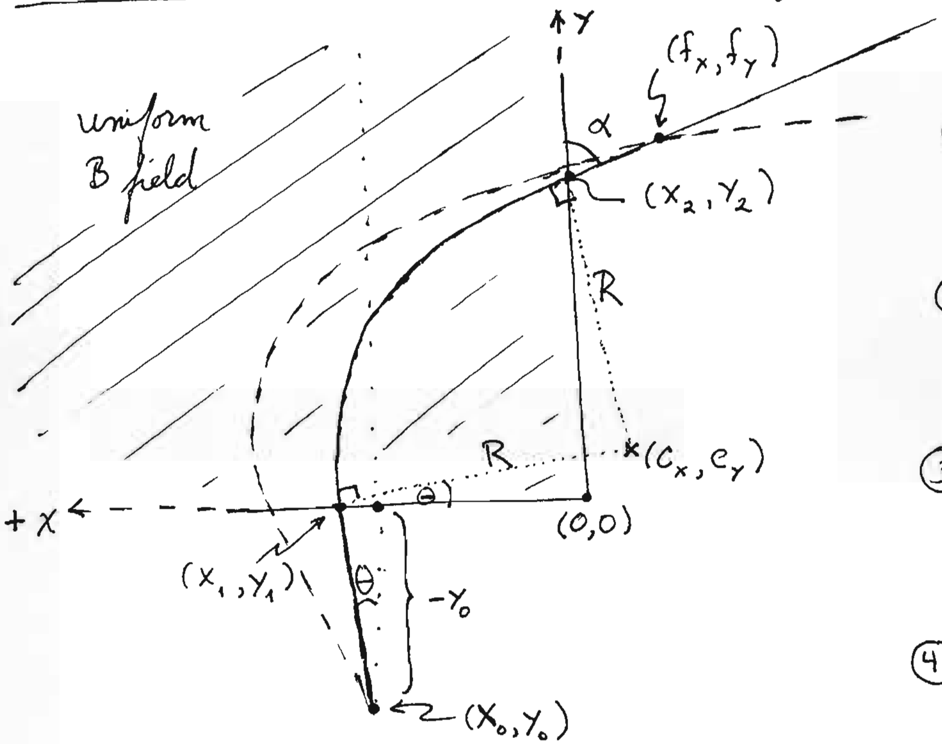
Focal properties of a rectangular dipole magnet - R. Jones

Aug. 2009



① $\begin{cases} X_1 = X_0 - Y_0 \tan\theta \\ Y_1 = 0 \end{cases}$

② $\begin{cases} C_x = X_1 - R\cos\theta \\ C_y = Y_1 + R\sin\theta \end{cases}$

③ $\begin{cases} (X_2 - C_x)^2 + (Y_2 - C_y)^2 = R^2 \\ X_2 = 0, \quad Y_2 = C_y + \sqrt{R^2 - C_x^2} \end{cases}$

④ $-2\,C_x\,dx + 2(Y_2 - C_y)\,dy = 0$

$$\cot\alpha = \frac{dy}{dx} = \frac{C_x}{Y_2 - C_y}$$

⑤ $$\frac{f_x\,d\alpha}{\sin\alpha_0} = dy\,\sin\alpha_0$$

$$\therefore\ f_x = \sin^2\alpha_0\,\frac{dy}{d\alpha}$$

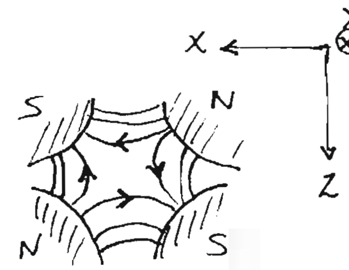
Everywhere in the first quadrant (right upper above) there is a uniform B field and everywhere outside that quadrant $B = 0$.

Implementation

In the file excel-fp.xls I implemented the above solution. The same basic formulas lead to results both for point-to-point and parallel-to-point focal planes. In either case, I evaluate the derivative $dy_2/d\alpha$ using finite difference methods.

* point-point: shift θ and track y_2 vs α
* parallel-point: shift x_0 and track y_2 vs α

Addition of the quadrupole field



At the right is shown a beam's eye view of the quadrupole field. It is measured by the quantity Q in Tesla/m. A negative Q is desired to focus in z.

$$\vec{B} = (+\hat{z}\,x + \hat{x}\,z)\,Q$$

length $= \ell$

For a particle moving in direction $\vec{v}$, it experiences the force

$$\dot{p}_x = q\,(v_y B_z - v_z \underbrace{\cancel{B_y}}_{0}), \quad \dot{p}_z = q\,(v_x \underbrace{\cancel{B_y}}_{0} - v_y B_x)$$

I neglect the quadrupole fringe field, but not that of the dipole. For the trajectory shown on page 1, and a quadrupole with its axis aligned on the nominal particle path, (using quadrupole coordinates, with y along its axis)

$$\dot{p}_x = q v x Q \quad , \quad \dot{p}_z = -q v z Q$$

$$\Delta P_x = q v x \int_0^{\ell} Q \, dt = \frac{q v x}{v} \int_0^{\ell} Q \, dy = q \ell Q x$$

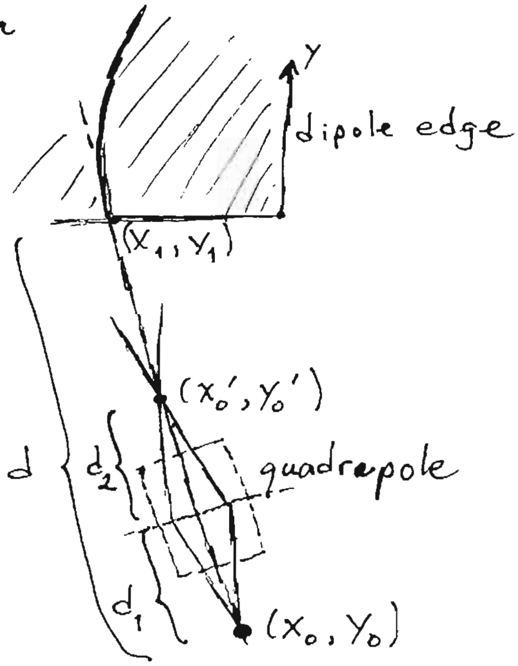
$$\therefore \quad \Delta P_z = - q \ell Q z$$

Both share a common focal length $|f| = \dfrac{P|z|}{\Delta P_\perp}$

$$f = \pm \frac{P}{q \ell Q} \quad (+ \text{ for } x, \ - \text{ for } z)$$

I ncorporate this result into the optics model above, using the usual optics formula

$$\frac{1}{f} = \frac{1}{d_1} + \frac{1}{d_2}$$



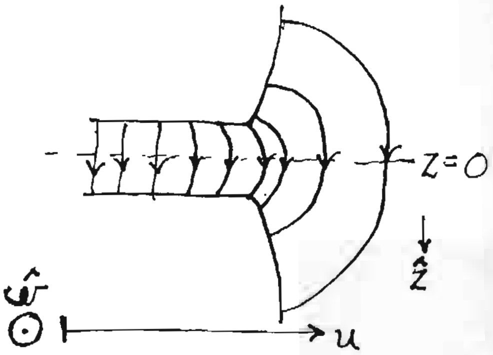
as shown in the figure, and the image at (X_0', Y_0') becomes the new source of rays going forward. A negative focal length leads to a negative d_2 as desired. For point-point optics the picture is as-shown.

For parallel-point optics, the value f is substituted for d_2.

Incorporating the dipole fringe field

The above treatment of the quadrupole applies equally in both xy and zy planes. To find the correct vertical focal properties, the fringe field of the dipole must also be included.

Consider the particle shown at the right moving in the fringe field of the dipole magnet



$$\dot{\vec{p}} = q\,\vec{v} \times \vec{B}$$

$$\vec{B}(z) \stackrel{\sim}{=} +B\hat{z} - B'\hat{u}\,z$$

Both B, B' are strong functions of u.

($\hat{u}$ is outward normal to the pole edge)

Furthermore let $v_z \approx 0$

$$\dot{p}_u = q(v_w B_z - \cancelto{0}{v_z B_u})$$

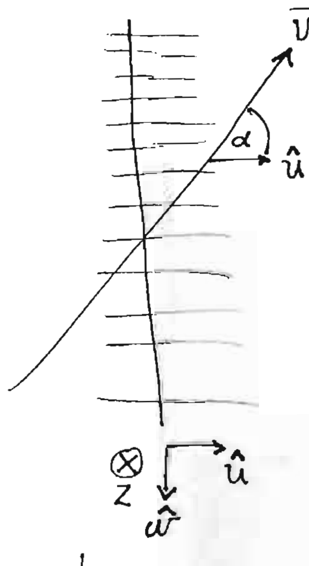
$$\dot{p}_z = q(\cancelto{0}{v_u B_w} - v_w B_u)$$

$$\dot{p}_w = q(\cancelto{0}{v_z B_u} - v_u B_z)$$

$\dot{p}_u = q v_w B$, $\dot{p}_w = -v_u B$ are the regular orbit equations for the particle near the boundary, and are already accounted for in the optics model. What is new is:

$$\dot{p}_z = +q v_w B' z$$

Now consider the net ΔP experienced by a particle going through the fringe field

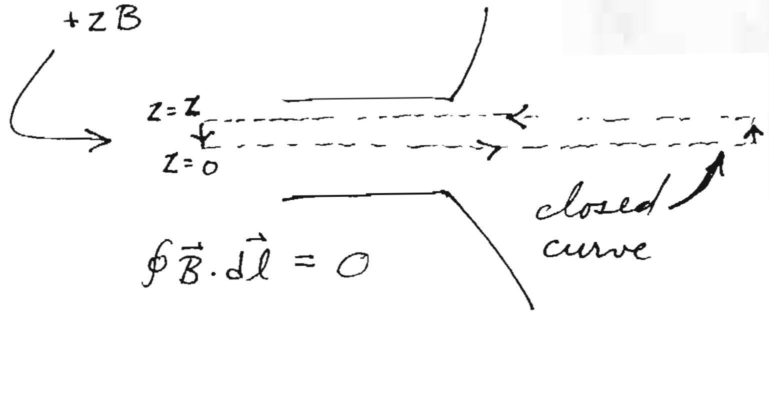


$$\Delta P_z = -\int q v B' \sin\alpha\, z \left(\frac{ds}{v}\right)$$

$$= -\frac{q \sin\alpha}{\cos\alpha} \underbrace{\int_{-\infty}^{\infty} zB'(u)\,du}_{+zB}$$

$$\Delta P_z = -qzB\tan\alpha$$

(generic result)



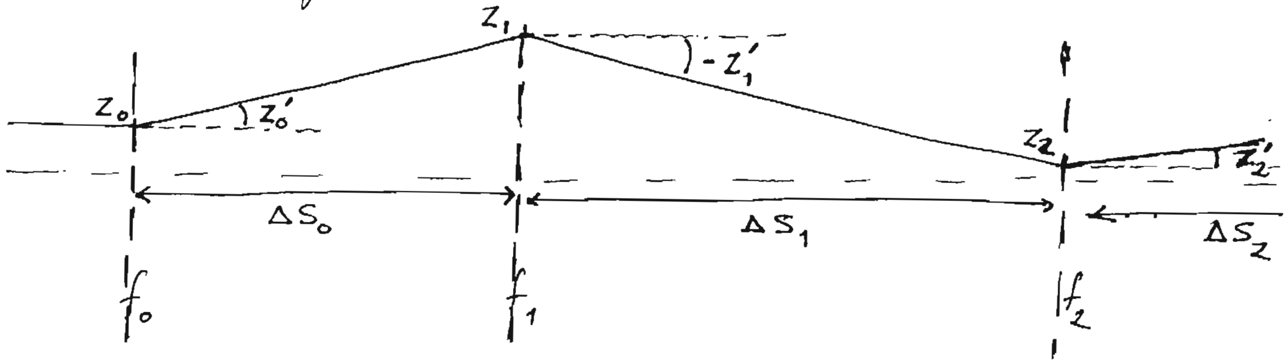
$$\oint \vec{B}\cdot d\vec{l} = 0$$

This applies directly to the exit edge of the magnet

$$\Delta P_{exit} = -qZB\cot\beta_0$$

$$\Delta P_{entry} = qZB\tan\theta$$

Consider a sequence of thin lenses spaced ΔS_i apart:



$$Z_{i+1} = Z_i + Z_i'\,\Delta S_i \quad , \quad Z_{i+1}' = Z_i' - \frac{Z_{i+1}}{f_i}$$

$$\begin{pmatrix} Z_{i+1} \\ Z_{i+1}' \end{pmatrix} = \begin{pmatrix} 1 & \Delta S_i \\ -\frac{1}{f_i} & 1-\frac{\Delta S_i}{f_i} \end{pmatrix} \begin{pmatrix} Z_i \\ Z_i' \end{pmatrix}$$

Everything is proportional to Z_0 so I replace it with 1 (in some units)

$$\begin{pmatrix} Z_n \\ Z_n' \end{pmatrix} = \begin{pmatrix} 1 & \Delta S_{n-1} \\ -\frac{1}{f_{n-1}} & 1-\frac{\Delta S_{n-1}}{f_{n-2}} \end{pmatrix} \begin{pmatrix} 1 & \Delta S_{n-2} \\ -\frac{1}{f_{n-2}} & 1-\frac{\Delta S_{n-2}}{f_{n-2}} \end{pmatrix} \cdots \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The vertical focus is achieved in the case where

$$Z_2 \cdot Z_2' < 0 \quad , \quad Z_2 > 0$$

at $\Delta S_2 = -\dfrac{Z_2}{Z_2'}$ ⇐ see figure in excel-fp.xls