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Tree-Level QED Cross Section for Triplet Production

$$\frac{d\sigma}{\text{"}d\Omega\text{"}} = \frac{(2\pi)^4}{\mathcal{F}_{in}} |M_{fi}|^2 \rho_f$$

where $\mathcal{F}_{in} = 4kmc$, k = lab energy of incident photon,
m = mass of electron.

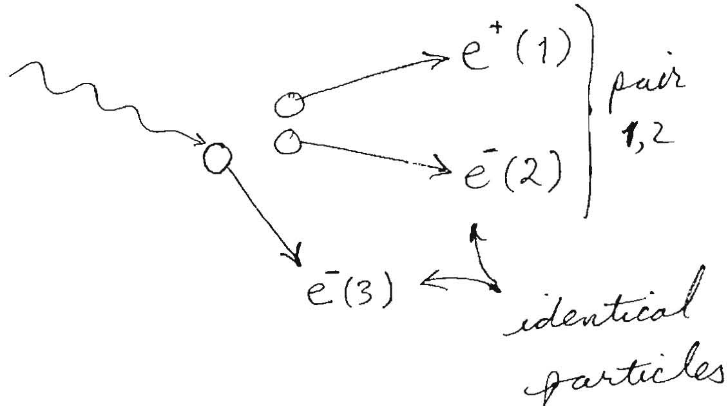
and

$$\rho_f = \left(\frac{1}{2\pi}\right)^9 \frac{d^3 q_1}{2E_1} \frac{d^3 q_2}{2E_2} \frac{d^3 q_3}{2E_3} \delta^4(q_0 - q_1 - q_2 - q_3)$$

where $q_0 = (k+m, \vec{k})$, $\vec{q}_1$ is momentum of final positron, q_2 and q_3 are momenta of the final electrons. Depending on how you want to write "$d\Omega$" in $\frac{d\sigma}{\text{"}d\Omega\text{"}}$, ρ_f can be written in many forms, and is 5-dimensional. Let the 5 dimensions be : (recoil) q_3^1, q_3^2, M_{12}, ϕ_{12}, E_2

- q_3^1, q_3^2: lab frame x, y components of recoil momentum
- M_{12}: invariant mass of pair 1, 2
- ϕ_{12}: azimuthal angle of q_2 in rest frame of pair 1, 2
- E_2: lab energy of final e^- 2

Example reaction diagram



To simplify ρ_f, make a change of variables in the forward pair 1,2 from q_1, q_2 to $q_{12} = q_1 + q_2$ and $k_{12} = \frac{1}{2}(q_1 - q_2)$

$$d^4q_1\, d^4q_2 = d^4q_{12}\, d^4k_{12}$$

$$\therefore \rho_f = \left(\frac{1}{2\pi}\right)^9 d^4q_1\, d^4q_2\, d^4q_3\, \delta(q_1^2-m^2)\,\delta(q_2^2-m^2)\,\delta(q_3^2-m^2)\,\delta^4(q_0-q_1-q_2-q_3)$$

$$= \left(\frac{1}{2\pi}\right)^9 d^4q_{12}\, d^4k_{12}\, d^4q_3\, \delta([\tfrac{q_{12}}{2}+k_{12}]^2-m^2)\,\delta([\tfrac{q_{12}}{2}-k_{12}]^2-m^2)\,\delta(q_3^2-m^2)\,\delta^4$$

but $d^4q_{12}\,\delta^4(q_0 - q_{12} - q_3) = 1$

$$\therefore \rho_f = \left(\frac{1}{2\pi}\right)^9 \frac{d^3q_3}{2E_3}\left\{ d^4k_{12}\,\delta([\tfrac{q_{12}}{2}+k_{12}]^2-m^2)\,\delta([\tfrac{q_{12}}{2}-k_{12}]-m^2)\right\}$$

$$\rightarrow dk_{12}^0\,\delta([\tfrac{q_{12}}{2}+k_{12}]^2-m^2) = \underbrace{dk_{12}^0\,\delta([\tfrac{m_{12}}{2} - k_{12}^0]^2 - |\vec{k}_{12}^*|^2 - m^2)}_{\text{in c.m. frame of the pair}}$$

$$= \frac{1}{2m_{12}} \text{ (in } \vec{q}_{12}=0 \text{ frame)}$$

$$\rightarrow d|\vec{k}_{12}|\,\delta([\tfrac{q_{12}}{2}-k_{12}]^2-m^2) = \underbrace{d|\vec{k}_{12}|\,\delta([\tfrac{m_{12}}{2}]^2 - |\vec{k}_{12}^*|^2 - m^2)}_{\text{in c.m. frame of the pair}}$$

$$= \frac{1}{2|\vec{k}_{12}^*|}$$

where $|\vec{k}^*| = \sqrt{\frac{m_{12}^2}{4} - m^2}$

$$\therefore \rho_f = \left(\frac{1}{2\pi}\right)^9 \frac{\sqrt{m_{12}^2 - 4m^2}}{8\, m_{12}\, E_3} \text{ on measure } d^3q_3\, d^2\Omega_{12}^*$$

(Ω_{12}^*:) c.m. direction of $\vec{q}_2$ in pair rest frame.

Alternately, one may express the z component of $\vec{q}_3$ as $f(m_{12}^2)$:

$$(q_0 - q_3)^2 = q_{12}^2 = m_{12}^2$$

$$(k + m - E_3)^2 - (\vec{k} - \vec{q}_3)^2 = m_{12}^2$$

$$= \cancel{k^2} + m^2 + \underline{E_3^2} + 2km - 2kE_3 - 2mE_3 - \cancel{k^2} - \underline{q_3^2} + 2k q_3^3$$

$$= 2m^2 + 2km - 2E_3 k - 2E_3 m + 2k q_3^3$$

$$= 2(k+m)(m - E_3) + 2k q_3^3 \quad , \quad E_3 = \sqrt{(q_3^3)^2 + (q_{3\perp})^2 + m^2}$$

$$dm_{12}^2 = \left(2k + 2(k+m)\left(-\frac{q_3^3}{E_3}\right)\right) dq_3^3$$

$$dq_3^3 = \frac{E_3}{2kE_3 - 2(k+m) q_3^3} dm_{12}^2 = \frac{E_3 m_{12}}{kE_3 - (k+m) q_3^3} dm_{12}$$

This conversion enables us to write "$d\Omega$" as either $d^3 q_3 \, d^2\Omega_{12}^*$ or as $d^2 q_{3\perp} \, d\phi_{12}^* \, dm_{12}$ and adjust ρ_f accordingly.

$$\rho_f' = \left(\frac{1}{2\pi}\right)^9 \frac{\sqrt{m_{12}^2 - 4m^2}}{8kE_3 - 8(k+m) q_3^3} \quad , \quad q_3^3 = \frac{m_{12}^2 - 2(k+m)(m - E_3)}{2k}$$

Furthermore, we can exchange $d\cos\theta^*$ for dE_2 in the lab.

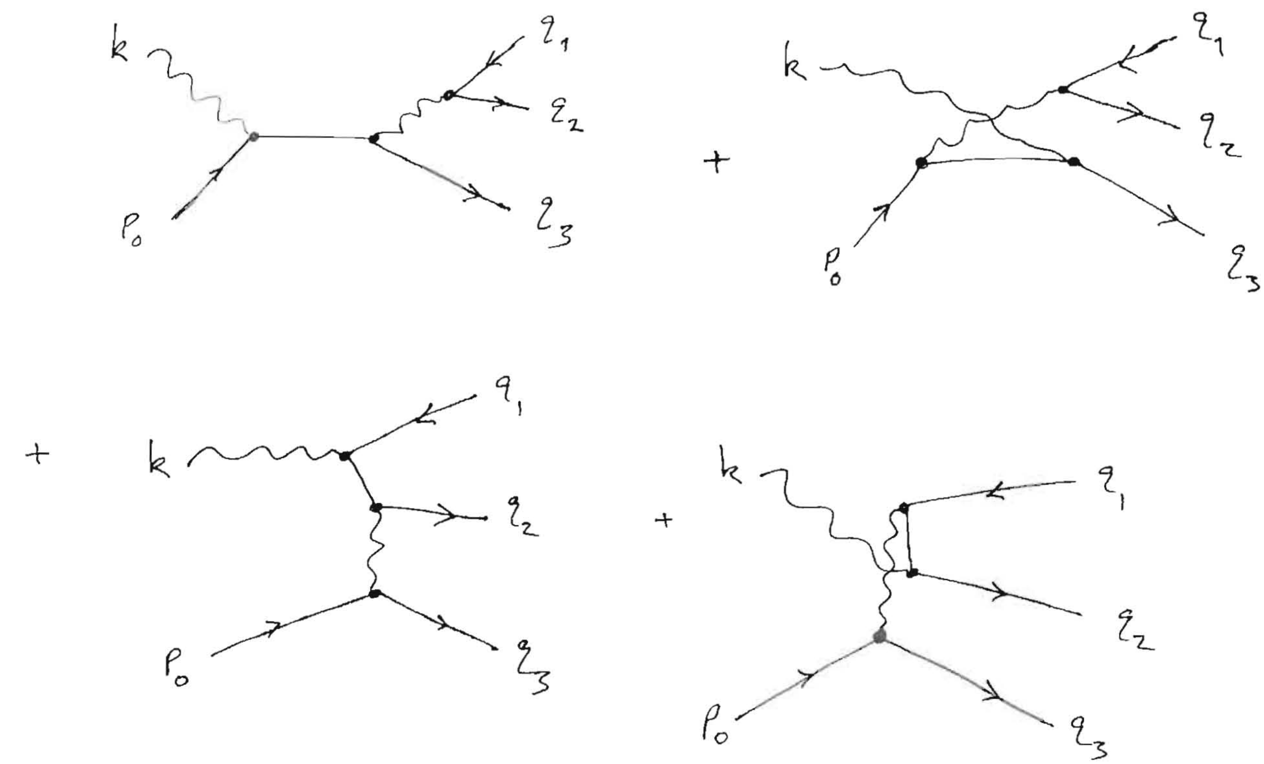
$$E_2 = \gamma \frac{m_{12}}{2} + \gamma\beta k_{12}^* \cos\theta_{12}^* \quad , \quad \gamma = \frac{E_{12}}{m_{12}} \, , \; \gamma\beta = \frac{|\vec{q}_{12}|}{m_{12}}$$

$$= \frac{E_{12}}{\cancel{m_{12}}} \frac{\cancel{m_{12}}}{2} + \frac{|\vec{q}_{12}|}{m_{12}} k_{12}^* \cos\theta_{12}^*$$

$$dE_2 = \frac{k_{12}^* |\vec{q}_{12}|}{m_{12}} d\cos\theta^*$$

$$\rho_f'' = \left(\frac{1}{2\pi}\right)^9 \frac{m_{12}}{8kE_3 - 8(k+m) q_3^3} \frac{1}{|\vec{q}_{12}|} \quad \text{on measure } d^2 q_{3\perp} \, d\phi_{12}^* \, dm_{12} \, dE_2$$

M_{fi} :



– 4 terms with $2 \leftrightarrow 3$ exchanged, <u>8 diagrams</u>

$$
\begin{aligned}
M_{fi} = \; & e^3 \bar{u}(q_2)\gamma^\mu v(q_1)\,\bar{u}(q_3)\gamma_\mu \frac{1}{(q_{12}^2)} \frac{1}{\not{k}+\not{p}_0-m} \not{\epsilon}\, u(p_0) \\
& + e^3 \bar{u}(q_2)\gamma^\mu v(q_1)\,\bar{u}(q_3)\not{\epsilon} \frac{1}{(q_{12}^2)} \frac{1}{\not{q}_3-\not{k}-m} \gamma_\mu u(p_0) \\
& + e^3 \bar{u}(q_2)\gamma^\mu \frac{1}{\not{k}-\not{q}_1-m} \not{\epsilon}\, v(q_1) \frac{1}{(p_0-q_3)^2} \bar{u}(q_3)\gamma_\mu u(p_0) \\
& + e^3 \bar{u}(q_2)\not{\epsilon} \frac{1}{\not{q}_2-\not{k}-m} \gamma^\mu v(q_1) \frac{1}{(p_0-q_3)^2} \bar{u}(q_3)\gamma_\mu u(p_0) \\
& - \{ 4 \text{ terms with } 2 \leftrightarrow 3 \text{ exchanged} \}
\end{aligned}
$$

$\Rightarrow$ computing $|M_{fi}|^2$ involves computing 64 terms (8×8) each one the trace of a product of 12 Dirac matrices!