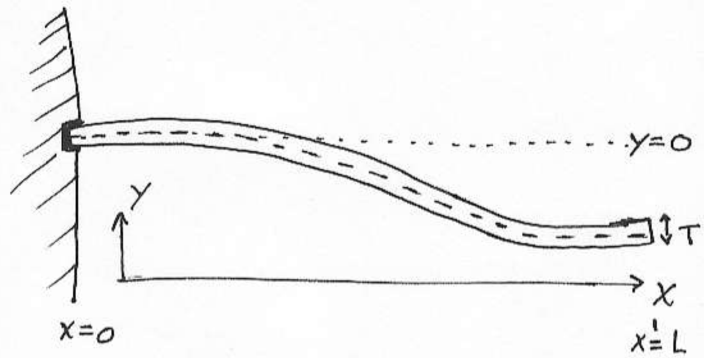
R. Jones

Q. What is the shape and frequency of a stiff sheet undergoing small-amplitude oscillations about a flat equilibrium geometry? Ignore gravity.

* Assumptions: the sheet is held fixed, with fixed slope, at one end, taken to be the left end. The right end is free, as shown below.

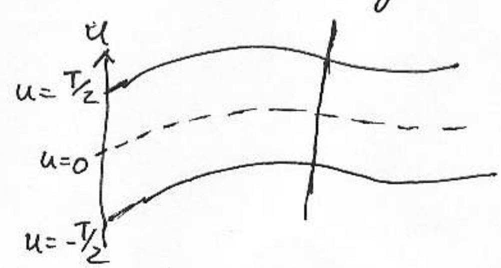
* Coordinate system: X, Y axes, with origin, shown in the figure to the right.



* Dimensions: the plank is of length L, thickness T, width W, where T and W functions of X and not necessarily constants. The plank is made of a uniform material of density ρ and bulk modulus E.

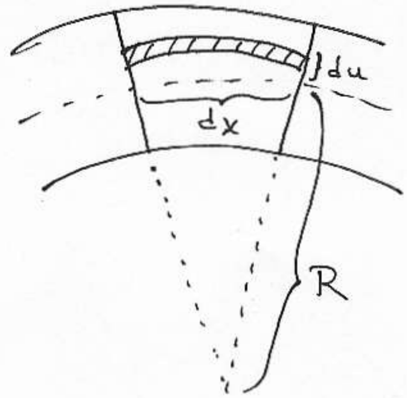
* Analysis of static forces at fixed displacement

Make an imaginary cut through the plank at X and ask what forces the part at the left places on that to the right



Let u measure depth inside the plank relative to the central layer. Forces are tension (F_x) and shear (F_y). F_x arises from bending of the material and its bulk elasticity. F_y arises from the fact that the plank remains a single continuous piece — it is a constraint in the problem.

To get F_x, consider a bent shape as shown to the right. The layer indicated by du is stretched from its equilibrium length dx by a distance



$$\Delta x = \left(\frac{R+u}{R}\right)dx - dx = \frac{u}{R}dx$$

What is R? It is given by the second derivative for $y(x)$

Let $y - y_0 = \frac{1}{2} y'' dx^2$, then $R^2 = dx^2 + (y_0 + \frac{1}{2} y'' dx^2)^2$

But $y_0 = R$, so $\cancel{R^2} = dx^2 + \cancel{R^2} + R y'' dx^2 + \Theta\left((dx^2)^2\right)$

$$\therefore \quad y'' = -\frac{1}{R}, \quad R = -\frac{1}{y''}$$

So $\Delta x = -u y'' dx$. What is the force element from du?

$$dF_x = E \frac{\Delta x}{dx} du\, W = -W E u y'' du$$

$$F_x = \int_{-T/2}^{T/2} \frac{dF_x}{du} du = -WEy''\left(\frac{u^2}{2}\right)\Big|_{-T/2}^{T/2} = -\frac{1}{8} WEy''(T^2 - T^2) = 0$$

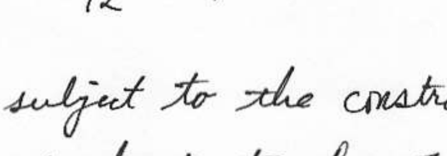
Forces cancel, but torques do not. Tension forces make torque

$$\Gamma = \int_{-T/2}^{T/2} \frac{dF_x}{du} u\, du = -WEy''\left(\frac{u^3}{3}\right)\Big|_{-T/2}^{T/2} = -\frac{1}{12} WEy'' T^3$$

To propagate this force along the plank, subject to the constraint that the plank remains intact, we need to kept track of both the shear $S(x)$ and torque $\Gamma(x)$ as we move along.

$$S(x - dx) = S(x) + \ddot{y}(x)\, dm$$

$$\Gamma(x - dx) = \Gamma(x) + S(x) dx$$



$dm = \rho W T dx$

$$\left.\begin{aligned} S' &= -\rho W T \ddot{y} \\ \Gamma' &= -S \end{aligned}\right\} \quad \Gamma'' = \rho W T \ddot{y}$$

assume "thin plank"

$L >> T$

$$E \frac{d^2}{dx^2}\left(W T^3 \frac{d^2 y}{dx^2}\right) = -\rho W T \frac{d^2 y}{dt^2}$$

This is not a wave equation because it is fourth order in X, but it can support wave-like solutions under certain conditions.

Case a) Let W, T be constants, then W drops out!

$$\frac{d^4 y}{dx^4} = -\left(\frac{\rho}{ET^2}\right)\frac{d^2 y}{dt^2}$$

Let $y = y_0 e^{i(kx-\omega t)}$, $k^4 y_0 = \frac{\rho\omega^2}{ET^2} y_0$

Boundary conditions are $y = y' = 0$ @ $x = 0$

$y'' = y''' = 0$ @ $x = L$

Normal mode condition: $y(x,t) = y(x)\, e^{i\omega t}$

$$\frac{d^4 y}{dx^4} = +\left(\frac{\rho}{ET^2}\right)\omega^2 y$$

$$y = y_1 e^{ikx} + y_2 e^{-ikx} + y_3 e^{kx} + y_4 e^{-kx}, \quad k = \left(\frac{\rho\omega^2}{ET^2}\right)^{1/4}$$

① $y_1 + y_2 + y_3 + y_4 = 0$

② $i(y_1 - y_2) + (y_3 - y_4) = 0$

③ $-y_1 e^{ikL} - y_2 e^{-ikL} + y_3 e^{kL} + y_4 e^{-kL} = 0$

④ $-iy_1 e^{ikL} + iy_2 e^{-ikL} + y_3 e^{kL} - y_4 e^{-kL} = 0$

These equations are homogeneous, so solutions only exist subject to certain constraints on k (or ω). This is what gives the eigenmode equation.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ i & -i & 1 & -1 \\ -e^{ikL} & -e^{-ikL} & e^{kL} & e^{-kL} \\ -ie^{ikL} & ie^{-ikL} & e^{kL} & -e^{-kL} \end{vmatrix} = 0$$

Let $a = e^{ikL}$, $b = e^{kL}$

$$= \left\{ -i\left[-1-1\right] - 1\left[\frac{1}{ab} - \frac{i}{ab}\right] - 1\left[-\frac{b}{a} - i\frac{b}{a}\right] \right\}$$

$$- \left\{ i\left[-1-1\right] - 1\left[\frac{a}{b} + i\frac{a}{b}\right] - 1\left[-ab + iab\right] \right\}$$

$$+ \left\{ i\left[\frac{1}{ab} - \frac{i}{ab}\right] + i\left[\frac{a}{b} + i\frac{a}{b}\right] - 1\left[-i-i\right] \right\}$$

$$- \left\{ i\left[-\frac{b}{a} - i\frac{b}{a}\right] + i\left[-ab + iab\right] + 1\left[-i-i\right] \right\}$$

$$= 8i + \frac{2i}{ab} + \frac{2ib}{a} + \frac{2ia}{b} + 2iab$$

$$\therefore \quad 4ab + 1 + b^2 + a^2 + a^2b^2 = 0$$

$$(a+b)^2 + (ab+1)^2 = 0$$

$\therefore$ $a = 1, b = -1$ or $a = -1, b = 1$, neither one is possible!

or a is complex! in fact, it is a pure phase $e^{i\phi_a}$

$(ab+1) = \pm i(a+b)$, b is pure real, $b > 1$.

$$\left.\begin{array}{ll} ① & a_r + b = \pm a_i b \\ ② & a_r b + 1 = \mp a_i \end{array}\right\} \quad \frac{a_r}{b} + a_r b + 2 = 0 \Rightarrow a_r = -\frac{2b}{b^2+1}$$

$$a_i = \pm \frac{1-b^2}{1+b^2}$$

$$\phi_a = \tan^{-1}\left(\frac{a_i}{a_r}\right) = \boxed{\tan^{-1}\left(\frac{b^2-1}{2b}\right) = \pm \ln b}$$