Influence of Readout Non-uniformity on Resolution in a Calorimeter

R.T. Jones
University of Connecticut, Storrs, CT

March 27, 2008

Abstract

The intrinsic energy resolution of a calorimeter is limited by sampling fluctuations and photon detection statistics. The reconstructed photon energy resolution actually achieved in an experiment is generally somewhat worse than the intrinsic resolution because of a number of factors, which include external energy loss through early conversions and energy leakage from the calorimeter, as well as various imperfections in the calorimeter response. Many of the latter effects can be mitigated through a careful offline calibration procedure, but those arising from variations in the readout sensitivity that occur between different segments within a single digitization channel cannot. This report examines the general problem of readout non-uniformities and their effect on calorimeter resolution, and shows that they contribute both to the statistical and floor terms in the usual energy resolution formula, with the dominant effects seen in the floor term. General formulas are derived that can be used to calculate the final calorimeter resolution resulting from a given distribution of sensitivities at different scales of segmentation. When formulas are applied to the specific case of the GlueX barrel calorimeter, the results show that the final resolution is surprisingly insensitive to large variations in the gain between individual SiPM tiles or Planacon anode segments, once the offline inter-channel gain calibration is carried out, provided that light guides with suitable properties are used.

The GlueX barrel calorimeter energy measurement is based on a sum of signals from many independent sensors. These sensors are summed in subgroups to form electronic signals for digitization. Each of these digitized signals is multiplied by its own gain constant and then summed over contiguous regions to generate the calorimeter energy measurement for a shower. The channel gain constants are not known a-priori, so they must be determined from the data using a calibration procedure. Offline calibration procedures are used by virtually all calorimeter experiments to determine these gain constants in a time-dependent fashion based on the raw data collected by the experiment. One such bootstrap procedure that has proved effective for high-energy photoproduction experiments is presented in Ref. [1]. Discussion of the methods involved is outside the scope of this note, except to point out that the best one can do using such a calibration procedure is to asymptotically approach the calorimeter resolution that is permitted by the intrinsic resolution plus other limiting factors. One of these other limiting factors is non-uniform gain that may occur within a segment of the calorimeter subtended by a single electronics channel. For example, a light guide may have a non-uniform probability of delivering a photon that enters its collection end to the surface of the phototube at the other end, and the phototube may have a non-uniform photoelectron efficiency across the face of its photocathode.

Such non-uniformities are present in every calorimeter, but they are of particular concern in the region of the GlueX barrel calorimeter being read out using silicon photomultiplier (SiPM) arrays. The SiPM arrays are composed of 3mm x 3mm tiles arranged in 4 x 4 groups and summed electronically to form a single digitization channel. Early prototypes of the tiled SiPM modules indicate that variations in the product of the photon detection efficiency (PDE) times the gain of individual tiles within a single module may be as large as a factor 2 or even larger, unless special measures are taken to match them. *Special measures always entail costs.* Therefore it is important to obtain quantitative information on the impact that fine-grained variations in the calorimeter sensitivity have on its energy resolution. This is the issue addressed in this report.

1   The Basic Model

Consider a calorimeter as a large array composed of many identical sampling elements. The sampling element functions as the calorimetric unit cell in the following analysis. In the GlueX barrel calorimeter it may be considered to be a single fiber together with the surrounding lead and epoxy that constitutes its unit cell. In a homogeneous calorimeter such as the lead-glass array, a sampling element may be taken to be a 1 mm^3 cube within a lead glass block. The details of how the unit cell is conceived is not important, provided that the following assumptions are satisfied.

1. There are many elements per readout channel.

2. There are many readout channels in the full calorimeter.

3. A single gain constant g_i describes the proportionality between the energy deposition E_i in cell i and its mean contribution $\langle y_i \rangle$ to the response measured in the readout channel to which the element belongs.

4. Apart from the different constants g_i, the response of all cells in the calorimeter to a given deposited energy E_i within that cell is the same.

Consider an ensemble of many showers, each of which deposits exactly the same total contained[1] energy E in the calorimeter. The showers are randomly distributed within the acceptance of the calorimeter, however that is defined. For each event in the sample,

$$E = \sum_{i=0}^{N} E_i \tag{1}$$

where N represents the total number of cells in the entire calorimeter, summed across all readout channels. Note that this is an exact relation, because E is the total contained energy and E_i is the local deposition in cell i. The $\{E_i\}$ vary from event to event in the sample, but E is the same for all events and is henceforth treated as a constant. To visualize this, the reader should note that most of the E_i are zero for a given event, and the non-zero values are mostly grouped in a limited region within the vicinity of the shower axis.

The intrinsic response of the calorimeter in the model is encapsulated in the distribution of the random variable $\hat{s}_i \sim R(E_i)$ where R depends in a non-trivial way on the energy deposition E_i. The details of this distribution are not important, however, but only its mean and its variance. These are defined as follows.

$$\langle \hat{s}_i \rangle = E_i \tag{2}$$

$$\left\langle \hat{s}_i{}^2 \right\rangle - \langle \hat{s}_i \rangle^2 = \alpha^2 E_i \tag{3}$$

where the constant α is the same for all cells. Eq. 2 merely defines the scale for random variable $\hat{s}_i{}^2$, whereas Eq. 3 has important physical content. The meaning of the constant α is clarified by evaluating the variance of the random variable $s = \sum_{i=1}^{N} s_i$ which represents the total energy measured by all readout channels across the calorimeter. Before it is possible to compute the variance of s, an expression is needed for the variance of the sample values s_i for cell i. The distribution of these values is a convolution of two fluctuations: shower deposition variations that cause the $\{E_i\}$ to be different for every event, and sampling errors that produce a different s_i for event even when E_i is the same. These two fluctuations are independent, so their covariances are zero.

$$\left\langle s_i^2 \right\rangle - \langle s_i \rangle^2 = \alpha^2 \langle E_i \rangle + \left(\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right) \tag{4}$$

whereas

$$\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle = \langle E_i E_j \rangle - \langle E_i \rangle \langle E_j \rangle \tag{5}$$

because the sampling fluctuations in different cells are uncorrelated. Note that the same is not true of the energy depositions, because the constraint of energy conservation requires them to be correlated. With these results in hand, one can now easily compute the mean and variance of the total measured energy s.

$$\langle s \rangle = E \tag{6}$$

---

[1]The final calorimeter energy uncertainty also contains a contribution from external energy loss (backsplash, leakage out the ends, early conversions) but these are independent from the gain non-uniformity contribution and may be considered separately.

[2]The hat notation in the random variable $\hat{s}_i$ is used to denote the abstract random variable that represents a population for any given event with depositions $\{E_i\}$. Henceforth one is concerned with the concrete values s_{ik} that the random variable takes on for various events k in the sample ensemble. In what follows, the event index k is dropped and expectation values $\langle .. \rangle$ of variables without hats are used to denote ensemble averages of the concrete values, evaluated in the large-N limit of ensemble size.

$$V(s) = \sum_{i=1}^{N}\left(\left\langle s_i^2\right\rangle - \left\langle s_i\right\rangle^2\right) + \sum_{i\neq j}^{N}\left(\left\langle s_i s_j\right\rangle - \left\langle s_i\right\rangle\left\langle s_j\right\rangle\right)$$

$$= \alpha^2\sum_{i=1}^{N}\left\langle E_i\right\rangle + \sum_{i=1}^{N}\left(\left\langle E_i^2\right\rangle - \left\langle E_i\right\rangle^2\right) + \sum_{i\neq j}^{N}\left(\left\langle E_i E_j\right\rangle - \left\langle E_i\right\rangle\left\langle E_j\right\rangle\right)$$

$$= \alpha^2 E \tag{7}$$

where use has been made of the fact that

$$V_E = \sum_{i=1}^{N}\left(\left\langle E_i^2\right\rangle - \left\langle E_i\right\rangle^2\right) + \sum_{i\neq j}^{N}\left(\left\langle E_i E_j\right\rangle - \left\langle E_i\right\rangle\left\langle E_j\right\rangle\right)$$

$$= 0 \tag{8}$$

by energy conservation. Eqs. 6-7 lead to the familiar result for the intrinsic calorimeter energy resolution.

$$\frac{\sigma_s}{\langle s\rangle} = \frac{\alpha}{\sqrt{E}} \tag{9}$$

Eq. 9 shows that α is nothing other than the coefficient of the statistical error term in the familiar expression for the intrinsic energy resolution of the calorimeter, and that in this idealized model there is no floor term[3]. For the GlueX barrel calorimeter, it has a numerical value close to $\alpha = 0.05$.

The variable s represents the total shower energy measured by the calorimeter for the ideal case where all elements contribute to the observed response with the same weight factor. Gain non-uniformities are introduced to the model by supposing that the readout produces a distorted sum y of the individual cell responses, which is similar to s except that each cell gets weighted by a gain factor g_i.

$$y = \sum_{i=1}^{N} y_i = \sum_{i=1}^{N} g_i s_i \tag{10}$$

Similar to what was done above for s, the moments of the y distribution are computed as follows.

$$\langle y\rangle = \sum_{i=1}^{N} g_i\left\langle s_i\right\rangle$$

$$= gE \tag{11}$$

where the new constant g represents a global gain constant that simply sets the scale for the response y and has nothing to do with calorimeter resolution.

$$g \equiv \frac{1}{E}\sum_{i=1}^{N} g_i\left\langle E_i\right\rangle \tag{12}$$

By simply rescaling y one may arbitrarily set $g = 1$. Now it is convenient to express the g_i in terms of gain mismatch variables δ_i.

$$g_i = 1 + \delta_i \tag{13}$$

which obey the relation

$$\sum_{i=1}^{N}\delta_i\left\langle E_i\right\rangle = 0 \tag{14}$$

---

[3]In is shown below how gain non-uniformities lead naturally to the appearance of a floor term.

as follows from Eq. 12.

These results may be collected to calculate the mean and variance of the y energy measurement.

$$\langle y \rangle = E \tag{15}$$

$$V(y) = \sum_{i=1}^{N} \left(\left\langle y_i^2 \right\rangle - \langle y_i \rangle^2 \right) + \sum_{i \neq j}^{N} \left(\langle y_i y_j \rangle - \langle y_i \rangle \langle y_j \rangle \right)$$

$$= \sum_{i=1}^{N} (1 + \delta_i)^2 \left(\left\langle s_i^2 \right\rangle - \langle s_i \rangle^2 \right) + \sum_{i \neq j}^{N} (1 + \delta_i)(1 + \delta_j) \left(\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \right)$$

$$= \sum_{i=1}^{N} (1 + \delta_i)^2 \left(\alpha^2 \langle E_i \rangle + \left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right)$$

$$+ \sum_{i \neq j}^{N} (1 + \delta_i)(1 + \delta_j) \left(\langle E_i E_j \rangle - \langle E_i \rangle \langle E_j \rangle \right)$$

$$= \alpha^2 E + 2\alpha^2 \sum_{i=1}^{N} \delta_i \langle E_i \rangle + \alpha^2 \sum_{i=1}^{N} \delta_i^2 \langle E_i \rangle$$

$$+ 2\sum_{i=1}^{N} \delta_i \left(\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right) + 2\sum_{i \neq j}^{N} \delta_i \left(\langle E_i E_j \rangle - \langle E_i \rangle \langle E_j \rangle \right)$$

$$+ \sum_{i=1}^{N} \delta_i^2 \left(\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right) + \sum_{i \neq j}^{N} \delta_i \delta_j \left(\langle E_i E_j \rangle - \langle E_i \rangle \langle E_j \rangle \right) \tag{16}$$

The assumption that there are a large number of cells and that the gain mismatch parameters average to zero allows us to neglect terms that contain a single power of the gain mismatch parameters δ_i because they cancel in the sum over many cells. This leaves only the intrinsic terms and terms containing two δ_i factors.

$$V(y) = \alpha^2 E + \alpha^2 \sum_{i=1}^{N} \delta_i^2 \langle E_i \rangle + \sum_{i=1}^{N} \delta_i^2 \left(\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right)$$

$$+ \sum_{i \neq j}^{N} \delta_i \delta_j \left(\langle E_i E_j \rangle - \langle E_i \rangle \langle E_j \rangle \right) \tag{17}$$

The final term proportional to $\delta_i \delta_j$ only sums to zero if the gain mismatches between different cells are really uncorrelated between the cells. This gives a physical significance to the choice of the size of the unit cells in the calorimeter that are used for computing results in the following sections. Choosing a calorimeter cell size that matches the scale of local variations in the gain mismatch allows one to consider the gains to be uniform within a cell, but uncorrelated between different cells. If the cell size is chosen smaller than this, then the last term in Eq. 17 will need to be included in computing $V(y)$. If the cell size is chosen larger than this, then the assumption that the gain is uniform within each cell is violated. Assuming that an appropriate choice for the cell size is made

so that the g_i are uncorrelated[4],

$$V(y) \quad = \quad \alpha^2 E + \alpha^2 \sum_{i=1}^{N} \delta_i^2 \langle E_i \rangle + \sum_{i=1}^{N} \delta_i^2 \left(\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right) \tag{18}$$

The physical interpretation of the terms in Eq. 18 is as follows. The first term proportional to α^2 is the intrinsic calorimeter energy resolution that includes both sampling fluctuations and photon detection statistics effects. The second term proportional to α^2 accounts for how the intrinsic uncertainties are amplified by gain non-uniformities[5]. The last term is typically more significant than the second. It reflects the fact that no two showers deposit energy the same way in the calorimeter, so each one is sampled by cells with different gains. Note that this term is independent of the intrinsic calorimeter resolution parameter α, and depends only on the segmentation of the calorimeter. Also note that it scales with E^2, so that it introduces a so-called *floor term* in the energy resolution formula

$$\frac{\sigma_y}{\langle y \rangle} = \frac{a}{\sqrt{E}} \oplus b \tag{19}$$

where a is the intrinsic resolution parameter α amplified by gain mismatch effects, and b is the floor term. The following sections are concerned with estimating the values of the parameters a and b.

2 A Concrete Case

To make the model more concrete, an ansatz is needed for the distribution of the gain mismatch parameters δ_i. Suppose they are distributed according to a quasi-normal distribution[6] with mean zero and standard deviation σ_δ, and that they are distributed among the cells in a way that is independent of the way that energy is typically shared within a shower. Within this statistical model for the gain mismatch distribution, Eq. 18 can be evaluated.

$$V(y) = \alpha^2 (1 + \sigma_\delta^2) E + \sigma_\delta^2 \sum_{i=1}^{N} \left(\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right) \tag{20}$$

The magnitude of the last term in Eq. 20 can be estimated by considering that variations in the depositions E_i come about principally because every shower hits the calorimeter in a different place.

Consider some two-dimensional coordinate $\vec{x}$ that measures the impact point of each gamma ray of energy E in the sample, as it enters the calorimeter. The energy E_i deposited in cell i by any given shower depends principally on the energy and position of the shower, and where the cell is located in the calorimeter (eg. front, middle, back). Let the coordinate $\vec{x_i}$ represent the value of shower coordinate $\vec{x}$ which maximizes the average value of E_i in cell i, and call that maximum

---

[4] Actually when offline gain corrections are applied at the level of readout channels, this introduces anti-correlations between the δ_i within a single readout channel, so that the term containing the product $\delta_i \delta_j$ in Eq. 17 has a negative sign that cancels some of the excess error from the last term in Eq. 18. This fact is ignored in Eq. 18, but is re-introduced later in the section entitled *Offline Calibration*

[5] As an extreme example, imagine that the gains are zero past a certain depth in the calorimeter. This can be compensated by increasing the global gain factor, but the sampling fluctuations in the parts that do produce a response are amplified by the larger gain factor.

[6] The normal distribution will not serve the purpose because gains must be positive definite, whereas normal random variables cover the entire real axis. A chi-square random variable with a suitably chosen number of degrees of freedom D has the advantage of being positive definite and asymptotically normal for large D. For this example, $\delta_i = (\hat{x}_i / D) - 1$ where $\hat{x}_i \sim \chi^2(D)$ so that the δ_i have zero mean and standard deviation $\sqrt{2/D}$. D was chosen to give the desired standard deviation σ_δ in the values for δ_i.

value ϵ_i. All other showers with different impact coordinates $\vec{x}$ produce reduced E_i values. Let the systematic variation of E_i with $\vec{x}$ be presented by the shower shape function $f(\vec{x})$ which is defined to have a maximum of 1 at $\vec{x} = 0$ and go to zero as $|\vec{x}| \to \infty$. Let $\mathcal{A}$ represent the finite area in $\vec{x}$ that describes the calorimeter acceptance. Furthermore assume that the function f falls off sufficiently fast as $|\vec{x}| \to \infty$ so that the integrals

$$
\begin{aligned}
f_0(i) &= \frac{1}{\mathcal{A}} \int d^2x \, \mathcal{I}(\vec{x}) \\
f_1(i) &= \frac{1}{\mathcal{A}} \int d^2x \, f(\vec{x} - \vec{x}_i) \, \mathcal{I}(\vec{x}) \\
f_2(i) &= \frac{1}{\mathcal{A}} \int d^2x \, f^2(\vec{x} - \vec{x}_i) \, \mathcal{I}(\vec{x})
\end{aligned}
\tag{21}
$$

are well-defined. Here $\mathcal{I}(\vec{x})$ represents the distribution of the shower sample across the acceptance of the calorimeter. The constants $f_n(i)$ depend on the shower spatial distribution but not on the shower energy E.[7] For a large sample, it follows that

$$
\begin{aligned}
\langle E_i \rangle &= \frac{\epsilon_i}{f_0(i)} f_1(i) \\
\left\langle E_i^2 \right\rangle &= \frac{\epsilon_i^2}{f_0(i)} f_2(i)
\end{aligned}
$$

so that

$$
\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 = \epsilon_i^2 \left(\frac{f_2(i)}{f_0(i)} - \frac{f_1^2(i)}{f_0^2(i)} \right) .
\tag{22}
$$

Note that the variance of the E_i is proportional to ϵ_i^2 independent of what is taken for the shower shape model $f(\vec{x})$. Since ϵ_i is proportional to the total shower energy E,[8] it follows that $Var(E_i)$ is proportional to E^2.

As a concrete example, consider the GlueX barrel calorimeter and a sample with fixed polar angle, so that $x \equiv \phi$ varies only in one dimension. Taking a smooth Gaussian model for the transverse profile of a shower, and assuming a uniform azimuthal shower distribution leads to the following expression for the deposition in cell i as a function of shower centroid position x.

$$
E_i = \epsilon_i e^{-\frac{(x-x_i)^2}{2r_E^2}}
\tag{23}
$$

where ϵ_i is the maximum energy deposited in cell i, which occurs when the shower axis passes through the cell, and r_E is the transverse rms radius of a shower of energy E. Such a simple one-dimensional model might apply to showers distributed uniformly in azimuthal angle ϕ in the GlueX barrel calorimeter. Averaging over the ensemble implies an average over x.

$$
\begin{aligned}
\langle E_i \rangle &= \frac{\epsilon_i}{L} \int_0^L e^{-\frac{(x-x_i)^2}{2r_E^2}} \, dx \\
&= \sqrt{2\pi} \frac{r_E}{L} \epsilon_i
\end{aligned}
\tag{24}
$$

---

[7]There are logarithmic corrections to this statement, which must be included if one wants a consistent description over many orders of magnitude. These introduce logarithmic corrections to the standard statistical + floor term energy resolution formula, and are beyond the scope of this study.

[8]*ditto*

where L represents the extent of generalized coordinate x that lies within the calorimeter acceptance.

$$
\begin{aligned}
\left\langle E_i^2 \right\rangle &= \frac{\epsilon_i^2}{L} \int_0^L e^{-\frac{(x-x_i)^2}{r_E^2}} \, dx \\
&= \sqrt{\pi} \frac{r_E}{L} \epsilon_i^2 \\
&= \frac{1}{\sqrt{2}} \left\langle E_i \right\rangle \epsilon_i
\end{aligned}
\tag{25}
$$

$$
\left\langle E_i^2 \right\rangle - \left\langle E_i \right\rangle^2 = \frac{1}{\sqrt{2}} \left\langle E_i \right\rangle \epsilon_i \left(1 - \sqrt{\pi} \frac{2r_E}{L} \right)
\tag{26}
$$

Summing this expression over all cells leads to the result

$$
V(y) = \alpha^2 (1 + \sigma_\delta^2) E + \sigma_\delta^2 E \frac{\bar{\epsilon}}{\sqrt{2}} \left(1 - \sqrt{\pi} \frac{2r_E}{L} \right)
\tag{27}
$$

An estimate for the last factor can be obtained in the case of the GlueX barrel calorimeter by estimating the transverse shower size to be one third of the width of one module, so that $2r_E/L = 1/144$. In a course-grained test of the formalism, one may divide the module into p cell layers in depth with layer thicknesses chosen to equalize the value of ϵ_i in each layer. In that case, one finds

$$
V(y) = \alpha^2 (1 + \sigma_\delta^2) E + \frac{\sigma_\delta^2}{\sqrt{2}p} E^2
\tag{28}
$$

for the GlueX barrel calorimeter. To test this, a Monte Carlo simulation of the GlueX barrel calorimeter was carried out consisting of 1,000,000 events with single showers of 1 GeV at polar angle 60° uniformly distributed in azimuth. The readout scheme used $p = 9$ and $\sigma_\delta = 0.15$, and $\alpha = 0$ in order to isolate the term of interest. The Monte Carlo result for $V(y)$ was 1.67×10^{-3}, as compared with 1.75×10^{-3} computed using the above model. Translated into calorimeter energy resolution, this corresponds to a floor term $b = 4.2\%$. This is a very large effect for such a modest rms gain mismatch of 15%, which demands the more refined treatment presented in the next section.

3   Offline Calibration

The simple uncorrelated model of random gain mismatches distributed to every cell in the calorimeter independently ignores the fact that these gain shifts are partially suppressed by gain corrections applied to each digitization channel in the offline data analysis. Gain constants are extracted from the data that allow the average gain mismatch in each readout channel to be reduced to near zero. Suppose that the cells are grouped into readout channels such that the set Q_r of cell indices identifies the cells that belong to readout channel r. Then

$$
\sum_{i \in Q_r} \delta_i = 0
\tag{29}
$$

for all readout channels r. In a similar manner to Eq. 8, this implies that

$$
\sum_{i \in Q_r} \delta_i^2 + {\sum_{i,j \in Q_r}}' \delta_i \delta_j = 0
\tag{30}
$$

where the prime on the second sum indicates that it excludes terms with $i = j$. In the limit of large N, the first sum in Eq. 30 can be replaced with $n_r \sigma_\delta^2$. Suppose that the δ_i have not been arranged in any special order, so that there are no additional correlations between them other than what is required by the constraint Eq. 29. In that case, Eq. 30 provides an expression for the mean value of $\delta_i \delta_j$ for pairs $i \neq j$, after the gains have been calibrated. This allows Eq. 18 to be corrected for the effects of offline gain calibration.

$$
\begin{aligned}
V(y) &= \alpha^2 E + \alpha^2 \sum_{i=1}^{N} \delta_i^2 \langle E_i \rangle + \sum_{i=1}^{N} \delta_i^2 \left(\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right) \\
&\quad + \sum_r \sideset{}{'}\sum_{i,j \in Q_r} \delta_i \delta_j \left(\langle E_i E_j \rangle - \langle E_i \rangle \langle E_j \rangle \right)
\end{aligned}
\tag{31}
$$

Within a single readout channel, the different E_i are highly correlated, whereas the δ_i are anti-correlated as required by Eq. 30. Thus the last sum in Eq. 31 has a negative sign and partially cancels the sum that precedes it. If the readout cells are smaller than the size of a typical shower then one may approximate the correlation in the E_i between different cells within a single readout channel as nearly perfect, in which case $\langle E_i E_j \rangle - \langle E_i \rangle \langle E_j \rangle \rightarrow \langle E_i^2 \rangle - \langle E_i \rangle^2$ and the last two terms in Eq. 31 completely cancel. However, in practice shower energy deposition is grainy and the correlations are never perfect. Examination of the Monte Carlo results reported in the previous section suggest that intra-channel correlations of the E_i are on the order of 80%-85% for cell sizes that correspond to the GlueX barrel calorimeter readout segmentation. Thus

$$
\begin{aligned}
V(y) &= \alpha^2 E + \alpha^2 \sum_{i=1}^{N} \delta_i^2 \langle E_i \rangle + (1 - \rho) \sum_{i=1}^{N} \delta_i^2 \left(\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right) \\
&= \alpha^2 (1 + \sigma_\delta^2) E + (1 - \rho) \sigma_\delta^2 E^2 \frac{1}{\sqrt{2}p}
\end{aligned}
\tag{32}
$$

where ρ is the correlation coefficient between E_i and E_j

$$
\rho_{ij} = \frac{\langle E_i E_j \rangle - \langle E_i \rangle \langle E_j \rangle}{\sqrt{\left[\langle E_i^2 \rangle - \langle E_i \rangle^2 \right] \left[\left\langle E_j^2 \right\rangle - \langle E_j \rangle^2 \right]}}
\tag{33}
$$

for cells $i \neq j$ within the same readout channel. In writing Eq. 32 it was assumed that $(\langle E_i^2 \rangle - \langle E_i \rangle^2) = (\left\langle E_j^2 \right\rangle - \langle E_j \rangle^2)$ for i, j in the same readout channel, and that ρ_{ij} is approximately the same for all i, j and all channels r, so that the subscripts can be dropped. The factor $(1 - 2\sqrt{\pi}\rho/L)$ in Eq. 27 has been dropped in Eq. 32 because it is within a few percent of unity for large calorimeters like those in GlueX, and the uncertainty in the factor $(1 - \rho)$ makes it irrelevant. The value $\rho = 0.80 \pm 0.05$ is a good estimate for the GlueX barrel calorimeter. The offline gain calibration reduces the size of the floor term of the resolution formula by a factor $\sqrt{1 - \rho} \approx 0.4$ relative to the case where no channel-by-channel gain match is done. For the case examined in the previous section, the model predicts $b = 1.8\%$ instead of $b = 4.1\%$ once the gain calibration is taken into account.

Of course, one might like to tolerate intra-channel gain mismatch factors significantly larger than $\sigma_\delta = 15\%$. Eq. 32 shows that the floor term is proportional to σ_δ. This means that relaxing the uniformity specification on σ_δ from 15% to 33% would bring the floor term back up the to level of the original estimate $b = 0.041$. A floor term of 4% is higher than is usually considered tolerable for electromagnetic calorimeters. An acceptable design goal for the GlueX barrel calorimeter might be $b \leq 0.02$, which corresponds to $\sigma_\delta \leq 0.17$. How this upper bound translates into specifications for the calorimeter readout components is the subject of the next section.

4   Light Guide Mixing

Naively it may appear that σ_δ refers to the standard deviation in the gain of the photon detector that is observed by scanning a point light source across its surface. This would be the case if the light collection system entailed a one-to-one map between cells in the calorimeter and segments of the photodetector surface, eg. if the fibers were directly coupled to the detectors. However in the GlueX barrel calorimeter there is a light guide between them, as is the case in most electromagnetic calorimeter designs. The primary purpose for the light guide is to reduce the required photodetector area, but it also has the important side effect of reducing the impact of local variations in the photodetector gain across its surface on the calorimeter energy resolution. The effects of light guide mixing on the energy resolution are taken into account as follows.

Consider that both the calorimeter and the photon detector each have n_r independent cells within readout channel r, and that the coupling between them allows any calorimeter cell to illuminate any detector tile. Let the coupling between the calorimeter cells and the photodetector tiles be described by a matrix C_{ij}, such that

$$y_i = \sum_{j=1}^N \gamma_j \, C_{ij} \, s_i \tag{34}$$

where γ_j is the exact analog of g_i in Eq. 10, except that it refers to the readout gain for light collected by tile j on the photodetector instead of for light produced in cell i. The coupling matrix C_{ij} is a N x N matrix in block-diagonal form, with positive definite elements located in n_r x n_r blocks along the diagonal. It follows from comparing Eq. 10 with Eq. 34 that

$$g_i = \sum_{j=1}^N C_{ij}\gamma_j \tag{35}$$

which shows that the effect of matrix C being different from the unit matrix is to make the $\{g_i\}$ more uniform than the $\{\gamma_j\}$. Suppose that the γ_j can be described by a quasi-normal distribution centered on 1 with some standard deviation σ_d, and that the gains γ_j fluctuate independently from tile to tile across the photodetector surface. Let

$$\gamma_j = 1 + d_j \tag{36}$$

for the $\{\gamma_j\}$ in analogy with Eq. 13 for the $\{g_i\}$. It is convenient to normalize the coupling matrix C_{ij} by the relations

$$\sum_{i,j\in Q_r} C_{ij} = n_r \tag{37}$$

for all readout channels r, consistent with the limiting case of the unit matrix $C_{ij} = \delta_{ij}$. The offline calibration guarantees that the net response in each readout channel r is correct, that is

$$\langle y_r \rangle = \sum_{i\in Q_r} \langle E_i \rangle \equiv \langle E_r \rangle \tag{38}$$

where E_r is the total energy deposition within the cells subtended by readout channel r. Eq. 38 can be rewritten in terms of the matrix C by assuming that the group of cells subtended by readout cell r is small enough that the $\langle E_i \rangle$ are approximately equal for all $i \in Q_r$.

$$\sum_{i,j\in Q_r} C_{ij}d_j \;\; = \;\; 0 \tag{39}$$

It is natural to suppose that the light-guide couplings of all of the readout channels in the calorimeter have the same physical structure. Then all of the n_r x n_r sub-matrices along the diagonal in C_{ij} are identical, and can be denoted by the lower-case matrix c_{ij}. This assumption is not essential to the following treatment, but it is adopted because it allows the some of the formulas to be written in a simplified way. None of the final results depend on all of the cells actually having an identical light-guide coupling scheme.

Eq. 38 implies that the total measured energy has the correct mean $\langle y \rangle = E$. The remaining task is to compute its variance. Beginning from Eq. 34, it may be written as

$$V(y) = \sum_{i,i'}^{N} \sum_{j,j'}^{N} C_{ij} C_{i'j'} \gamma_j \gamma_{j'} \left(\langle s_i s_{i'} \rangle - \langle s_i \rangle \langle s_{i'} \rangle \right) \tag{40}$$

where all four indices i, i', j, j' sum from 1 to N, but the factors C_{ij} and $C_{i'j'}$ guarantee that only pairs i, j and i', j' within the same readout channel actually contribute to the sum. Eqs. 4-5 allows this to be rewritten as

$$V(y) = \sum_{i,i'}^{N} \sum_{j,j'}^{N} C_{ij} C_{i'j'} \gamma_j \gamma_{j'} \left(\alpha^2 \langle E_i \rangle \delta_{ii'} + \langle E_i E_{i'} \rangle - \langle E_i \rangle \langle E_{i'} \rangle \right) \tag{41}$$

Replacing $\langle E_i \rangle$ with $\langle E_r \rangle / n_r$ leads to

$$
\begin{aligned}
V(y) &= \alpha^2 \frac{1}{n_r} \sum_r \langle E_r \rangle \sum_{i \in Q_r} \sum_{j,j'} C_{ij} C_{ij'} (1 + d_j)(1 + d_{j'}) \\
&\quad + \sum_{i,i'}^{N} \sum_{j,j'}^{N} C_{ij} C_{i'j'} \gamma_j \gamma_{j'} \left(\langle E_i E_{i'} \rangle - \langle E_i \rangle \langle E_{i'} \rangle \right) \\
&= \alpha^2 E \, M_1 + \alpha^2 \frac{1}{n_r} \sum_r \langle E_r \rangle \sum_{i \in Q_r} \sum_{j,j'} C_{ij} C_{ij'} d_j d_{j'} \\
&\quad + \sum_{i,i'}^{N} \sum_{j,j'}^{N} C_{ij} C_{i'j'} (1 + d_j)(1 + d_{j'}) \left(\langle E_i E_{i'} \rangle - \langle E_i \rangle \langle E_{i'} \rangle \right)
\end{aligned} \tag{42}
$$

where in the last step terms have been dropped that involve a single power of the factors d_j because they average to zero. The factor M_1 introduced in Eq. 42 is one of several bilinear sums of the c_{ij} that are introduced to abbreviate the formulas.

$$M_0 = \frac{1}{n_r} \sum_{i=1}^{n_r} \sum_{j=1}^{n_r} c_{ij}^2 \tag{43}$$

$$M_1 = \frac{1}{n_r} \sum_{i=1}^{n_r} \sum_{j=1}^{n_r} \sum_{j'=1}^{n_r} c_{ij} c_{ij'} \tag{44}$$

$$M_2 = \frac{1}{n_r} \sum_{i=1}^{n_r} \sum_{i'=1}^{n_r} \sum_{j=1}^{n_r} c_{ij} c_{i'j} \tag{45}$$

All three parameters M_0, M_1, and M_2 are normalized to equal 1 for the special case $c_{ij} = \delta_{ij}$.

The statistical model for the gain mismatch parameters d_j allows the remaining sums in Eq. 42 to be evaluated in the large-N limit. Each of them contain products of the form $d_j d_{j'}$. The d_j are assigned randomly to the individual tiles of the readout, without regard to what the C_{ij} are for

that tile or what the $\langle E_i \rangle$ values are for the cells that couple to it. Because of that, their values can be replaced with their averages in the large-N limit. The only restriction that ties them together for different j is Eq. 39 that expresses the overall channel gain match constraint of the offline gain calibration. If j and j' belong to different readout channels then their d_j and $d_{j'}$ are completely uncorrelated, in which case they average to zero just like the factors with a solitary d_j do. However terms with $j = j'$ do not average to zero because it is given that $\sum_{j=1}^{N} d_j^2 \rightarrow N\sigma_d^2$ in the large N limit.

$$\overline{d^2} = \sigma_d^2 \tag{46}$$

Squaring Eq. 39 can be used to obtain an average value $\overline{dd'}$ for $d_j d_{j'}$ with $j \neq j'$ within the same readout cell.

$$
\begin{aligned}
0 &= \sum_{i \in Q_r} \sum_{i' \in Q_r} \sum_{j,j'} C_{ij} C_{i'j'} d_j d_{j'} \\
&= \sum_{i \in Q_r} \sum_{i' \in Q_r} \sum_{j} C_{ij} C_{i'j} d_j^2 + \sum_{i \in Q_r} \sum_{i' \in Q_r} \sum_{j \neq j'} C_{ij} C_{i'j'} d_j d_{j'} \\
&= n_r \sigma_d^2 M_2 + \overline{dd'} \left[\left(\sum_{i,j \in Q_r} C_{ij} \right) \left(\sum_{i',j' \in Q_r} C_{i'j'} \right) - n_r M_2 \right] \\
\overline{dd'} &= -\frac{\sigma_d^2 M_2}{n_r - M_2} \tag{47}
\end{aligned}
$$

Substitution of Eqs. 46-47 into Eq. 42 leads to

$$
\begin{aligned}
V(y) &= \alpha^2 E\, M_1 + \alpha^2 E \left[\sigma_d^2 M_0 + \overline{dd'}(M_1 - M_0) \right] \\
&\quad + \sum_{i,i'}^{N} \sum_{j,j'}^{N} C_{ij} C_{i'j'} (1 + d_j)(1 + d_{j'}) \left(\langle E_i E_{i'} \rangle - \langle E_i \rangle \langle E_{i'} \rangle \right) \\
&= \alpha^2 E\, M_1 + \alpha^2 E\, \sigma_d^2 \left[M_0 - \frac{M_2(M_1 - M_0)}{n_r - M_2} \right] \\
&\quad + \sum_{i,i'}^{N} \sum_{j,j'}^{N} C_{ij} C_{i'j'} (1 + d_j)(1 + d_{j'}) \left(\langle E_i E_{i'} \rangle - \langle E_i \rangle \langle E_{i'} \rangle \right) \\
&= \alpha^2 E \left[M_1 + \sigma_d^2 \left(\frac{n_r M_0 - M_1 M_2}{n_r - M_2} \right) \right] \\
&\quad + (1 - \rho) \sum_{i=1}^{N} \sum_{jj'}^{N} C_{ij} C_{ij'} d_j d_{j'} \left(\langle E_i^2 \rangle - \langle E_i \rangle^2 \right) \\
&\quad + \sum_{i,i'}^{N} \sum_{j,j'}^{N} C_{ij} C_{i'j'} \left(\langle E_i E_{i'} \rangle - \langle E_i \rangle \langle E_{i'} \rangle \right) \tag{48}
\end{aligned}
$$

Further work is required to evaluate the two remaining sums in Eq. 48. The first can be simplified by substitution of the average values for the products d_j^2 and $d_j d_{j'}$.

$$
\begin{aligned}
&\sum_{i=1}^{N} \sum_{jj'}^{N} C_{ij} C_{ij'} d_j d_{j'} \left(\langle E_i^2 \rangle - \langle E_i \rangle^2 \right) \\
&= \sum_{i=1}^{N} \sum_{jj'}^{N} C_{ij} C_{ij'} \left[\sigma_d^2 \delta_{jj'} - \left(\frac{\sigma_d^2 M_2}{n_r - M_2} \right) (1 - \delta_{jj'}) \right] \left(\langle E_i^2 \rangle - \langle E_i \rangle^2 \right)
\end{aligned}
$$

$$= \sigma_d^2 \left(M_0 - \left(\frac{M_2}{n_r - M_2} \right) (M_1 - M_0) \right) \sum_{i=1}^N \left(\left\langle E_i^2 \right\rangle - \left\langle E_i \right\rangle^2 \right)$$

$$= \sigma_d^2 \left(\frac{n_r M_0 - M_1 M_2}{n_r - M_2} \right) \sum_{i=1}^N \left(\left\langle E_i^2 \right\rangle - \left\langle E_i \right\rangle^2 \right) \tag{49}$$

To aid in the evaluation of the second sum in Eq. 48, it is helpful to introduce the new parameters e_i defined as

$$\sum_{j=1}^N C_{ij} = 1 + e_i \tag{50}$$

which measure the deviation from uniform net coupling between cell i and the calorimeter readout. Eq. 37 requires that the e_i average to zero across a readout channel.

$$\sum_{i,i'}^N \sum_{j,j'}^N C_{ij} C_{i'j'} \left(\langle E_i E_{i'} \rangle - \langle E_i \rangle \langle E_{i'} \rangle \right)$$

$$= \left\langle \left[\sum_{i=1}^N (1 + e_i) (E_i - \langle E_i \rangle) \right]^2 \right\rangle$$

$$= \sum_{i,i'}^N e_i e_{i'} \left(\langle E_i E_{i'} \rangle - \langle E_i \rangle \langle E_{i'} \rangle \right)$$

The last step above follows from Eq. 8. The terms $i \neq i'$ in this sum contain factors $e_i e_{i'}$ which have both signs and average to zero for large N, leaving only the terms with $i = i'$, so that

$$\sum_{i,i'}^N \sum_{j,j'}^N C_{ij} C_{i'j'} \left(\langle E_i E_{i'} \rangle - \langle E_i \rangle \langle E_{i'} \rangle \right)$$

$$= \sum_i^N e_i^2 \left(\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right)$$

$$= \left(\frac{1}{N} \sum_i^N e_i^2 \right) \sum_i^N \left(\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right)$$

$$= (M_1 - 1) \sum_i^N \left(\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right) \tag{51}$$

Substitution of these results back into Eq. 48 leads to a final result for the variance of the measured energy for a sample of showers with fixed contained energy E.

$$V(y) = \alpha^2 E \left[M_1 + \sigma_d^2 \left(\frac{n_r M_0 - M_1 M_2}{n_r - M_2} \right) \right]$$

$$+ (1 - \rho) \left[M_1 - 1 + \frac{n_r M_0 - M_1 M_2}{n_r - M_2} \right] \sum_{i=1}^N \left(\left\langle E_i^2 \right\rangle - \langle E_i \rangle^2 \right) \tag{52}$$

Comparison with Eq. 32 allows shows that this result agrees with the basic model, with the identification

$$\sigma_\delta^2 = M_1 - 1 + \sigma_d^2 \left(\frac{n_r M_0 - M_1 M_2}{n_r - M_2} \right) \tag{53}$$

Eq. 53 shows how specific design choices for the light-guides and photodetector uniformity translate into a floor term in the energy resolution formula.

5 Examples

Some examples of how to use the results derived in this report are useful to illuminate their meaning. In all of the following examples, fixed values are taken for the intrinsic calorimeter resolution parameter $a = 0.050$ and shower auto-correlation parameter $\rho = 0.080$.

5.1 Direct coupling

In the direct coupling example $C_{ij} = \delta_{ij}$, corresponding to photodetectors that are glued directly to the ends of the fibers, with 100% coverage. In this case, Eqs. 43-45 give $M_0 = M_1 = M_2 = 1$ and $\sigma_\delta = \sigma_d$. This is a worst-case scenario in the sense that variability in the local photodetector sensitivity translates directly into variability in the calorimeter sensitivity to local shower energy deposition fluctuations.

5.2 Perfect light guides

At the other extreme from direct coupling is the limit where the light guides couple all fibers in a readout channel to the photodetector surface with equal efficiency, but there is no correlation between which fiber produced the photon and which photodetector tile collected it. In this case all of the rows of the matrix c_{ij} are identical, ie. c_{ij} does not depend upon i, leading to $M_1 = 1$ and $M_0 = M_2$, which gives $\sigma_\delta = 0$. This means that there is no floor term at all for perfect light guides within this model, no matter how non-uniform the photodetectors turn out to be. Of course, this assumes that the offline gain calibration can be done perfectly, which is an idealization.

Note that in order to be *perfect* in this model, the light guides do not need to illuminate the photodetector uniformly – c_{ij} can still depend on photodetector tile index j – but they do need to collect light with uniform efficiency at the collection end and there can be no gaps between the light guides. Eqs. 52-53 provide a quantitative means to evaluate a concrete design to check how close it comes to the ideal coupling limit, in terms of the final energy resolution.

6 Summary

The final energy resolution of a calorimeter with a segmented photodetector readout is given by

$$V(y) = \alpha^2(1 + \sigma_\delta^2)E + (1 - \rho)\sigma_\delta^2 E^2 \frac{1}{\sqrt{2}p} \tag{54}$$

where y is the total measured energy after calibrated gain corrections have been applied, α is the intrinsic calorimeter resolution for 1 GeV showers, ρ is the shower deposition auto-correlation between independent sampling cells within one readout channel, and p is the number of independent sampling cells that lie within the core of a shower using energy-weighted counting with the maximum cell counting as 1.

The size of an independent sampling cell is set by the self-correlation of the readout efficiency described by the light coupling matrix C. For the GlueX calorimeter readout a conservative scale is obtained by dividing each readout light guide into 4 zones of roughly equal area (corners, edges, center, and mid-zone) which characterize the coupling across the entrance face of the light guide. Taking 4 independent cells per light guide and approximately 9 light guides for the energy-weighted

channel count for a 1 GeV shower[9] gives $p = 36$. Taking the value $\rho = 0.80$ leads to

$$V(y) = \alpha^2 (1 + \sigma_\delta^2)E + (0.063\,\sigma_\delta)^2\,E^2 \qquad (55)$$

for this readout geometry. The value of σ_δ is given by Eq. 53, with parameters M_0, M_1, and M_2 defined in Eqs. 43-45. How large a tolerance σ_d can be allowed for intra-channel gain variations within a photodetector module depends on the uniformity of light guide coupling described by the matrix C and encapsulated in the sums M_0, M_1, and M_2.

In the worst-case scenario where the photodetectors are glued directly onto the fibers, one obtains $\sigma_\delta = \sigma_d$. In this case, a specification $\sigma_d = 0.25$ would lead to a floor term in the energy resolution of $b = 1.6\%$, which might be considered acceptable. A more realistic light-guide scenario would lead to a significantly less stringent specification for the photodetector gain variability, for a given floor term.

7   Recommendations

Based on the results of the above study, it is recommended that a ray-tracing Monte Carlo study be made of the light-coupling efficiency function of the light guides to be used with the GlueX barrel calorimeter. A 4 x 4 grid on the input and on the output faces might be used, leading to a 16 x 16 matrix C. The parameters M_0, M_1, and M_2 defined in Eqs. 43-45 can then be used to evaluate the rms spread in local readout sensitivity σ_δ for a given rms spread in local photodetector gain σ_d. These results can then be used to derive a specification for σ_d that can be given to the photodetector manufacturer. If this specification results in significant extra production costs, the design of the light guide might be refined to further reduce σ_δ for fixed σ_d, so that the specification on σ_d can be further relaxed.

References

[1] R.T. Jones et.al., "A Bootstrap Method for Gain Calibration and Resolution Determination of a Lead-Glass Calorimeter", Nucl. Instr. Meth. **A 566** (2006) 366.

---

[9]The value of p depends on shower energy, but only weakly like $log(E/E_C)$, leading to logarithmic corrections to the standard energy resolution formula Eq. 19. These are not considered here.